36

THE SPHERE

You must have played or seen students playing football, basketball or table tennis. Football, basketball, table tennis ball are all examples of geometrical figures which we call "spheres" in three dimensional geometry.

If we consider the rotation of a semi-circle OAPB about its diameter AB, the rotation generates a sphere whose centre is the centre of the semi-circle and whose radius is equal to the radius of the semi-circle.

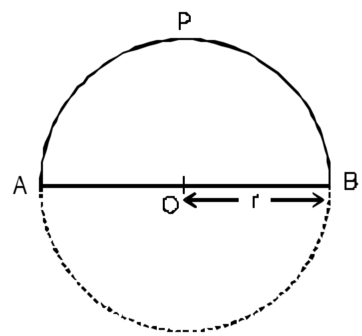

Fig. 36.1

Thus, a sphere is the locus of a point in space which moves in such a way that its distance from a fixed point, in space, always remains constant. The fixed point is called the **centre** of the sphere and the fixed distance is called the **radius** of the sphere.

The difference between a sphere and a circle is that a sphere is a figure in three-dimensional space while a circle is a figure in two dimensions (in a plane).

In this lesson, we shall study the equation of a sphere in centre-radius form, equation of a sphere through four non-coplanar points, the equation of a sphere in diameter form, plane section of a sphere and general equation of a sphere through a given circle.

OBJECTIVES

After studying this lesson, you will be able to find:

- the equation of a sphere in centre-radius form;
- the equation of a sphere in general form;
- the equation of a sphere through four non-coplanar points;
- the equation of a sphere in diameter form;
- the equation of a plane section of a sphere; and
- the general equation of a sphere through a given circle.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of two-dimensional coordinate geometry.
- Knowledge of three-dimensional geometry.
- Various forms of the equations of a plane.
- Straight line in space.

36.1 EQUATION OF A SPHERE IN CENTRE RADIUS FORM

Recall that a sphere is the set of points equidistant from a fixed point. The fixed point is called the **centre** of the sphere and the constant (or fixed) **distance** is its radius.

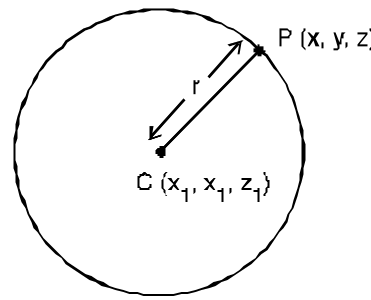

Fig. 36.2

Let P (x, y, z) be a point on the sphere whose centre is $C(x_1, y_1, z_1)$. Let r be the radius of the sphere.

$\therefore \quad CP^2 = r^2$ .....(i)

Using distance formula, we can write (i) as

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2 \quad .....(A)$$

which is the equation of the sphere in **centre-radius form.**

Corollary : If the centre of the sphere is at the origin, the equation of sphere with radius r is

$$x^2 + y^2 + z^2 = r^2 \quad ....(ii)$$

Note : The equation (A) is an equation of second degree in x, y and z.

We observe that :

(a) The co-efficients of the terms involving x^2, y^2 and z^2 are all equal (in this case each is equal to 1).

(b) There are no terms involving the products xy, yz or zx.

Thus, you will see that a general equation of second degree in x,yand z will represent a sphere if it satisfies the above two conditions.

(c) Consider an equation of the form

$$ax^2 + ay^2 + az^2 + 2lx + 2my + 2nz + d = 0 \qquad (a \neq 0) \quad .....(iii)$$

On dividing throughtout by 'a', equation (iii) can be written as

$$x^2 + y^2 + z^2 + \frac{2l}{a}x + \frac{2m}{a}y + \frac{2n}{a}z + \frac{d}{a} = 0 \quad .....(B)$$

(B) can be written in the form $x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0$ .....(iv)

where $g = \frac{l}{a}$, $f = \frac{m}{a}$, $h = \frac{n}{a}$ and $c = \frac{d}{a}$.

Equation (iv) can be written as

$$(x+g)^2+(y+f)^2+(z+h)^2 = \left(g^2+f^2+h^2-c\right) \quad .....(v)$$

Comparing (v) with (A) above, we have centre of the sphere (iv) as $(-g,-f,-h)$ and radius of the sphere (iv) as $\sqrt{g^2+f^2+h^2-c}$

Equation (iv) is called the general form of the equation of the sphere.

In order that the sphere may be real $g^2+f^2+h^2-c \geq 0$.

(d) In case r = 0, the sphere is a point sphere.

(e) The sphere whose centre is same as the centre of (iv) is called a concentric to sphere in (iv).The equation of the sphere concentric with (iv) is $x^2+y^2+z^2+2gx+2fy+2hz+k=0$ where k is a constant and this can determined from some other condition.

36.1.1 Interior and Exteroir of Sphere

Let O be the centre of a sphere with radius r. A point P_1 lies in the interior of the sphere if $OP_1 < r$. The point P_2 lies on the sphere if $OP_2 = r$ and a point P_3 lies in the exterior of the sphere if $OP_3 > r$.

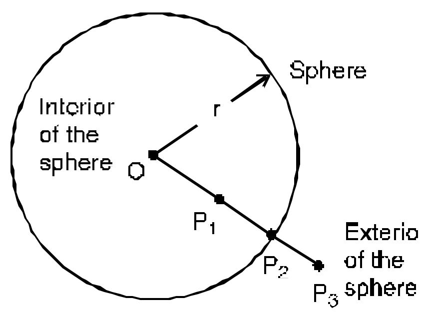

Fig. 36.3

Example 36.1 Find the equation of the sphere with centre at origin and radius 4.

Solution : The required equation of the sphere is

$$(x-0)^2+(y-0)^2+(z-0)^2=4^2$$

or $\quad x^2+y^2+z^2=16$

Example 36.2 Find the equation of the sphere with centre at $(2,-3,1)$ and radius $\sqrt{7}$.

Solution : The required equation of the sphere is

$$(x-2)^2+[y-(-3)]^2+(z-1)^2=\left(\sqrt{7}\right)^2$$

or $\quad (x-2)^2+(y+3)^2+(z-1)^2=7$

or $\quad x^2-4x+4+y^2+6y+9+z^2-2z+1-7=0$

or $\quad x^2+y^2+z^2-4x+6y-2z+7=0$

Example 36.3 Find the centre and radius of the sphere whose equation is

$$2x^2+2y^2+2z^2-4x+8y-6z-19=0$$

Solution : The given equation of the sphere is

$$2x^2+2y^2+2z^2-4x+8y-6z-19=0$$

or $$x^2+y^2+z^2-2x+4y-3z-\frac{19}{2}=0$$

$\therefore$ Here $$g=-1,\ f=2,\ h=-\frac{3}{2},\ c=-\frac{19}{2}$$

$\therefore$ Centre $$=(-g,-f,-h)=\left(1,\ -2,\ \frac{3}{2}\right)$$

and radius $$=\sqrt{g^2+f^2+h^2-c}=\sqrt{(-1)^2+(2)^2+\left(-\frac{3}{2}\right)^2+\frac{19}{2}}$$

$$=\sqrt{1+4+\frac{9}{4}+\frac{19}{2}}=\frac{\sqrt{67}}{2}$$

Example 36.4 Find the equation of the sphere which has its centre at the origin and which passes through the point (2,3,6).

Solution : We are given the centre of the sphere as (0, 0, 0).

$\because$ The point (2,3,6) lies on the sphere.

$\therefore$ Radius of the sphere = Distance of the origin from the point (2,3,6)

$$=\sqrt{(2-0)^2+(3-0)^2+(6-0)^2}$$

$$=\sqrt{4+9+36}$$

$$=\sqrt{49}$$

$$=7$$

$\therefore$ Equation of the required sphere is

$$(x-0)^2+(y-0)^2+(z-0)^2=7^2$$

or $$x^2+y^2+z^2=49$$

Example 36.5 For the sphere $x^2+y^2+z^2-2x+4y-6z-2=0$, find if the point (2,3,4) lies in the interior or exterior of the sphere.

Solution : The equation of given sphere is

$$x^2+y^2+z^2-2x+4y-6z-2=0.$$

Here $$g=-1,\ f=2,\ h=-3 \text{ and } c=-2$$

$\therefore$ Centre of the sphere $=(1,-2,3)$

Radius of the sphere $=\sqrt{1+4+9+2}=4$

The distance of the point (2,3,4) from centre

$$= \sqrt{(2-1)^2 + (3+2)^2 + (4-3)^2} = 3\sqrt{3}.$$

As $3\sqrt{3} > 4$, the point (2,3,4) lies in the exterior of the given sphere.

CHECK YOUR PROGRESS 36.1

1. Find the equation of the sphere whose centre is at the origin and whose radius is 5.
2. Find the centre and radius of the sphere

$$3x^2 + 3y^2 + 3z^2 - 3x + 6y - 9z - 17 = 0$$

3. Find the equation of the sphere which passes through the origin and

   (i) has the centre at the point $(3, -3, -1)$.

   (ii) has the centre at the point $(2, -2, -1)$.

4. Find the equation of the sphere which has its centre at the point $(3, -3 -1)$ and which passes through the point $(5, -2, 1)$
5. For the sphere $x^2 + y^2 + z^2 - 6x + 8y - 2z + 1 = 0$, find if the following points lie in the exterior, interior or on the sphere

   (i) $(2, -3, 4)$ (ii) $(-1, -4, -2)$ (iii) $(-1, 2, 3)$

36.2 EQUATIONS OF A SPHERE THROUGH FOUR NON-COPLANAR POINTS

Recall that the general equation of a sphere is

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0 \qquad .....(i)$$

This contains four constants g, f, h and c. If somehow we are able to determine the values of these constants, we can determine the equation of the sphere.

If it is given that the sphere passes through four non-coplanar points, it will give us four equations which will enable us to evaluate the four constants.

Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) be four non-coplanar points. These points will satisfy equation (i) as they lie on it.

$$\therefore \quad x_1^2 + y_1^2 + z_1^2 + 2gx_1 + 2fy_1 + 2hz_1 + c = 0 \qquad .....(1)$$

$$x_2^2 + y_2^2 + z_2^2 + 2gx_2 + 2fy_2 + 2hz_2 + c = 0 \qquad .....(2)$$

$$x_3^2 + y_3^2 + z_3^2 + 2gx_3 + 2fy_3 + 2hz_3 + c = 0 \qquad .....(3)$$

$$x_4^2 + y_4^2 + z_4^2 + 2gx_4 + 2fy_4 + 2hz_4 + c = 0 \qquad .....(4)$$

Solving (1), (2), (3) and (4) for g, f, h and c and substituting in (i), we get the required equation of sphere as

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_2^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Note : The student is not expected to evaluate the above determinant.

Example 36.6 Find the equation of the sphere passing through the points (0, 0, 0) (1, 0, 0), (0, 1, 0) and (0, 0, 1). Find also its radius.

Solution : Let the equation of the sphere be

$$x^2+y^2+z^2+2gx+2fy+2hz+c=0 \quad .....(i)$$

Since it passes through the point (0, 0, 0)

$\therefore \quad c = 0$

Again (i) pass through the point (1, 0, 0)

$\therefore \quad 1+0+0+2g+0+0+0+0=0$ or $g = -\frac{1}{2}$ [As c = 0]

Similarly, since it passes through the points (0, 1, 0) and (0, 0, 1)

we have, $f = -\frac{1}{2}$ and $h = -\frac{1}{2}$

$\therefore$ The equation (i) reduces to $x^2+y^2+z^2-x-y-z=0$

which is the required equation of the sphere.

Radius of the sphere $= \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-0} = \frac{\sqrt{3}}{2}$.

Example 36.7 Find the equation of the sphere which passes through the origin and the points $(2,1,-1)$, $(1,5,-4)$ and $(-2,4,-6)$. Find its centre and radius.

Solution : Let the equation of the sphere be

$$x^2+y^2+z^2+2gx+2fy+2hz+c=0 \quad .....(i)$$

Since it passes through the point (0, 0, 0), c = 0

Again (i) passes through the point $(2,1,-1)$

$\therefore \quad 2^2+1^2+(-1)^2+4g+2f-2h=0$

or $\quad 6 + 4g + 2f - 2h = 0$

or $\quad 3 + 2g + f - h = 0 \qquad$ .....(ii)

Similarly, as it passes through the points (1, 5, – 4) and (–2, 4, – 6)

We have have $42 + 2g + 10f - 8h = 0 \qquad$ .....(iii)

$$56 - 4g + 8f - 12h = 0 \qquad \text{.....(iv)}$$

Substituting the value of h from (ii) in (iii) and (iv) we get,

$$42 + 2g + 10f - 8(3 + 2g + f) = 0$$

or $\quad 18 - 14g + 2f = 0$

or $\quad 9 - 7g - f = 0 \qquad$ .....(v)

and $\quad 56 - 4g + 8f - 12(3 + 2g + f) = 0$

or $\quad 20 - 28g - 4f = 0$

or $\quad 5 - 7g - f = 0 \qquad$ .....(vi)

Solving (v) and (vi) for g and f, we get $g = 1,\ f = -2$.

Putting g = 1 and $f = -2$ in (ii), we get h = 3.

∴ The required equation of the sphere is

$$x^2 + y^2 + z^2 + 2x - 4y + 6z = 0$$

Centre of the sphere is $(-1, 2, -3)$.

and radius $= \sqrt{(1)^2 + (-2)^2 + 3^2 - 0} = \sqrt{14}$

Example 36.8 Find the equation of the sphere which passes through the points $(2,3,0)$, $(3,0,2)$ $(0,1,3)$ and $(2,2,0)$.

Solution : The required equation of the sphere is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 2^2+3^2+0^2 & 2 & 3 & 0 & 1 \\ 3^2+0^2+2^2 & 3 & 0 & 2 & 1 \\ 0^2+1^2+3^2 & 0 & 1 & 3 & 1 \\ 2^2+2^2+0^2 & 2 & 2 & 0 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 13 & 2 & 3 & 0 & 1 \\ 13 & 3 & 0 & 2 & 1 \\ 10 & 0 & 1 & 3 & 1 \\ 8 & 2 & 2 & 0 & 1 \end{vmatrix} = 0$$

CHECK YOUR PROGRESS 36.2

1. Find the equation of the sphere which passes through the four non-coplanar points given below :

   (i) (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c)

   (ii) (0, 0, 0), (–a, b, c), (a, – b, c) and (a, b, – c)

   (iii) (0, 0, 0), (0, 2, – 1), (–1, 1, 0) and (1, 2, – 3)

   Find the centre and radius of each of the above spheres obtained.

2. Find the equation of the sphere passing through the points $(1,-1,-1)$, $(3, 3, 1)$, $(-2, 0, 5)$ and $(-1, 4, 4)$.

36.3 DIAMETER FORM OF THE EQUATION OF A SPHERE

Let the point $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the extremities of a diameter of a sphere with centre O. Let $P(x, y, z)$ be any point on the sphere.

$\therefore$ PA and PB are at right angles to each other.

The direction ratios of PA and PB are $(x - x_1)$, $(y - y_1)$, $(z - z_1)$ and $(x - x_2)$, $(y - y_2)$, $(z - z_2)$ respectively.

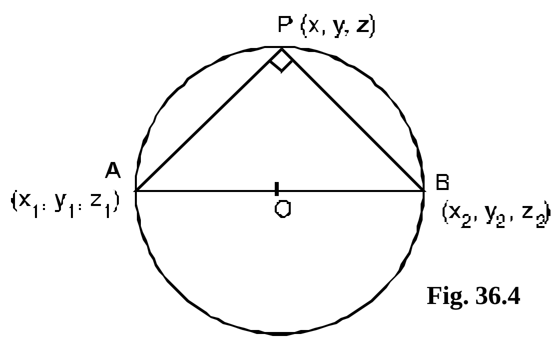

Fig. 36.4

Since PA and PB are at right angles.

$$\therefore (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0 \quad \text{....(i)}$$

which is the equation of the sphere in ***diameter form***

Note : The equation (i) can be rewritten as

$$x^2 + y^2 + z^2 - (x_1 + x_2)x - (y_1 + y_2)y - (z_1 + z_2)z$$
$$+ x_1x_2 + y_1y_2 + z_1z_2 = 0 \quad \text{....(A)}$$

Let us try to find the equation of the sphere by an ***alternative method***

As O is the mid-point of AB

$\therefore$ The co-ordinates of O are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

and radius of the sphere is

$$\frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$\therefore$ The equation of sphere is

$$\left[x - \frac{(x_1 + x_2)}{2}\right]^2 + \left[y - \frac{(y_1 + y_2)}{2}\right]^2 + \left[z - \frac{(z_1 + z_2)}{2}\right]^2$$

$$= \frac{1}{4}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]$$

$$= \left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2 + \left(\frac{z_2 - z_1}{2}\right)^2$$

which on simplification gives

$$x^2 + y^2 + z^2 - (x_1 + x_2)x - (y_1 + y_2)y - (z_1 + z_2)z$$

$$+\left[\left(\frac{x_2 + x_1}{2}\right)^2 - \left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 + y_1}{2}\right)^2 - \left(\frac{y_2 - y_1}{2}\right)^2\right.$$

$$\left.+\left(\frac{z_2 + z_1}{2}\right)^2 - \left(\frac{z_2 - z_1}{2}\right)^2\right] = 0$$

or $\quad x^2 + y^2 + z^2 - (x_1 + x_2)x - (y_1 + y_2)y - (z_1 + z_2)z$

$$+\frac{1}{4}(4x_1x_2 + 4y_1y_2 + 4z_1z_2) = 0$$

or $\quad x^2 + y^2 + z^2 - (x_1 + x_2)x - (y_1 + y_2)y - (z_1 + z_2)z$

$$+(x_1x_2 + y_1y_2 + z_1z_2) = 0$$

which is the same as (A) above.

Example 36.9 Find the equation of the sphere having extremities of one of its diameter as the points (2,3, 5) and $(-4, 7, 11)$. Find its centre and radius also.

Solution : The required equation of the sphere is

$$(x-2)(x+4) + (y-3)(y-7) + (z-5)(z-11) = 0$$

or $\quad x^2 + y^2 + z^2 + 2x - 10y - 16z + 68 = 0$

Centre of the sphere is $(-1, 5, 8)$ and radius

$$= \sqrt{(-1)^2 + 5^2 + 8^2 - 68} = \sqrt{22}.$$

Example 36.10 One end of a diameter of the sphere $x^2 + y^2 + z^2 - 2x - 6y - 2z + 2 = 0$ is the point $(3, 4, -1)$. Find the other end of the diameter.

Solution : Let the other end of the diameter be the point (x_1, y_1, z_1).

$\therefore$ The equation of the sphere described on the join of two given points (x_1, y_1, z_1) and $(3, 4, -1)$ is

$$(x - x_1)(x-3) + (y - y_1)(y-4) + (z - z_1)(z+1) = 0$$

or $\quad x^2 + y^2 + z^2 - (x_1 + 3)x - (y_1 + 4)y - (z_1 - 1)z$

$$+3x_1 + 4y_1 - z_1 = 0 \qquad .....(i)$$

Now equation (i) is identical with the given sphere

$$x^2 + y^2 + z^2 - 2x - 6y - 2z + 2 = 0 \qquad .....(ii)$$

Comparing the co-efficients of each terms of the equation (ii)

We have, $\quad x_1 + 3 = 2 \quad$ or $\quad x_1 = -1$

$y_1 + 4 = 6 \quad$ or $\quad y_1 = 2$

$z_1 - 1 = 2 \quad$ or $\quad z_1 = 3$

and $\quad 3x_1 + 4y_1 - z_1 = 2 \qquad .....(iii)$

$x_1 = -1,\ y_1 = 2$ and $z_1 = 3$ satisfy (iii) also.

$\therefore$ The co-ordinates of the other end of the diameter are $(-1, 2, 3)$.

Example 36.11 Find the centre and radius of the sphere

$$(x-2)(x-4) + (y-1)(y-3) + (z-2)(z+3) = 0$$

Solution : The equation of the sphere is

$$(x-2)(x-4) + (y-1)(y-3) + (z-2)(z+3) = 0$$

or $\quad x^2 - 6x + 8 + y^2 - 4y + 3 + z^2 + z - 6 = 0$

or $\quad x^2 + y^2 + z^2 - 6x - 4y + z + 5 = 0$

Here, $\quad g = -3,\ f = -2$ and $h = \frac{1}{2}$.

$\therefore \quad$ Centre of the sphere is $\left(3, 2, -\frac{1}{2}\right)$ and radius is

$$\sqrt{(3)^2 + (2)^2 + \left(-\frac{1}{2}\right)^2 - 5}$$

or $\quad \sqrt{9 + 4 + \frac{1}{4} - 5} \quad$ or $\quad \frac{\sqrt{33}}{2}$

CHECK YOUR PROGRESS 36.3

1. Find the equation of a sphere whose extremities of one of the diameter are

   (i) $(2, -3, 4)$ and $(-5, 6, -7)$

   (ii) $(2, -3, 4)$ and $(-1, 0, 5)$

(iii) $(5,4,-1)$ and $(-1,2,3)$

Also find the centre and radius of each of the above spheres.

2. If one end of a diameter of the sphere

$x^2+y^2+z^2-2x+4y-6z-7=0$ be the point $(-1,2,4)$, find the other end of the diameter.

3. Find the centre and radius of the sphere

$(x+1)(x+2)+(y-3)(y-5)+(z-7)(z-3)=0$

36.4 PLANE SECTION OF A SPHERE AND A SPHERE THROUGH A GIVEN CIRCLE

(i) Let us first consider the case of plane section of a sphere.

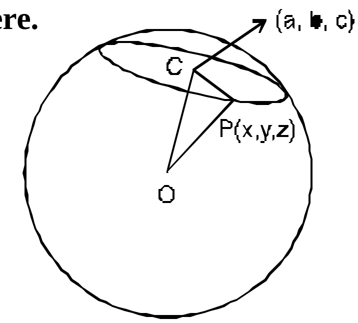

Fig. 36.5

We know that the equation of the sphere with origin as centre and radius r is given by $x^2+y^2+z^2=r^2$ .....(i)

Let C (a, b, c) be the centre of the plane section of the sphere whose equation we have to find out. The line segment OC drawn from O to the plane through C (a,b,c) is normal to the plane. The direction ratios of the normal OC are a, b, c.

Let P(x,y,z) be any point on the plane section.

The direction ratio of PC are $x-a,\ y-b, z-c$ and it is perpendicular to OC.

Using the condition of perpendicularity, we have

$$(x-a)a+(y-b)b+(z-c)c=0 \qquad ....(ii)$$

Equation (ii) is satisfied by the co-ordinates of any point P on the plane. Hence (i) and (ii) together constitute the equation of the plane section of the sphere.

Corollary 1 : If the centre of the sphere is $(-g,-f,-h)$, then the direction ratios of the normal OC are a + g, b + f, c + h.

Hence $(x-a)(a+g)+(y-b)(b+f)+(z-c)(c+h)=0$ .....(A)

and the equation of sphere is $x^2+y^2+z^2+2gx+2fy+2hz+d=0$ .....(B)

(A) and (B) together, in this case, constitute the equation of plane section of the sphere.

Corollary 2 : The plane section of a sphere by a plane passing through the centre of the sphere is called a **Great Circle**. You can see that

(i) the centre of this section coincides with the centre of the sphere. You may note that a great circle is a circle with greatest radius amongest all the plane section of the sphere.

(ii) the radius of the great circle is equal to the radius of sphere.

Now we consider the case of the sphere through a given circle.

(1) Let us consider the sphere

$$S \equiv x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0 \qquad .....(i)$$

and the plane $L \equiv lx + my + nz + k = 0$ .....(ii)

Equations (i) and (ii) together represent the equation of a circle, being the intersection of S = 0 and L = 0.

The equation $S + \lambda L = 0$, ...(iii)

where λ is a constant, gives the equation of the sphere passing through the circle given by (i) and (ii) together, for $S + \lambda L = 0$ is satisfied by the co-ordinates of the points lying on the circle.

Example 36.12 Find the centre and radius of the circle given by the equations

$$x^2 + y^2 + z^2 - 6x - 4y + 12z - 36 = 0, \; x + 2y - 2z = 1$$

Solution : Let $S \equiv x^2 + y^2 + z^2 - 6x - 4y + 12z - 36 = 0$ .....(i)

$L \equiv x + 2y - 2z - 1 = 0$ .....(ii)

(i) and (ii) together represents the equation of a circle.

We have, $(\text{radius of the circle})^2 = (\text{radius of the sphere})^2 - (\text{perpendicular distance from the centre upon the plane})^2$

The co-ordinates of the centre of (i) are $(3, 2, -6)$

$$\therefore \text{ Radius} = \sqrt{3^2 + 2^2 + (-6)^2 + 36} = \sqrt{85}$$

Perpendicular distance from the centre $(3, 2, -6)$ upon the plane (ii)

$$= \frac{1.3 + 2.2 - 2(-6) - 1}{\sqrt{1^2 + 2^2 + 2^2}} = 6$$

$\therefore$ $(\text{Radius of the circle})^2 = 85 - 36 = 49$ or, radius = 7

The equation

$$x^2 + y^2 + z^2 - 6x - 4y + 12z - 36 + \lambda(x + 2y - 2z - 1) = 0$$

represents a sphere passing through (i) and (ii)

The above equation can be rewritten as

$$x^2 + y^2 + z^2 - (6 - \lambda)x - (4 - 2\lambda)y + (12 - 2\lambda)z - 36 - \lambda = 0 \qquad ....(iii)$$

Now, the centre of the sphere given in (iii) is $\left(\frac{6-\lambda}{2}, 2 - \lambda, \lambda - 6\right)$

and radius is $\sqrt{\left(\frac{6-\lambda}{2}\right)^2 + (2-\lambda)^2 + (\lambda - 6)^2 + 36 + \lambda}$

$$\therefore \quad \left(\frac{6-\lambda}{2}\right)^2 + (2-\lambda)^2 + (\lambda - 6)^2 + 36 + \lambda = 7^2$$

or $36-12\lambda+\lambda^2+4(4-4\lambda+\lambda^2)+4(\lambda^2-12\lambda+36)-4(36+\lambda)=196$

or $9\lambda^2-72\lambda+144=0$

or $\lambda^2-8\lambda+16=0$

or $(\lambda-4)^2=0$

or $\lambda=4,4$

$\therefore$ The co-ordinates of the centre are $\left(\frac{6-4}{2},\,2-4,4-6\right)$ i.e., $(1,-2,-2)$.

Hence, the required radius and centre of the circle are 7 and $(1,-2,-2)$ respectively.

Example 36.13 Find the equation of the sphere for which the circle given by

$x^2+y^2+z^2+7y-2z+2=0$ and $2x+3y+4z-8=0$ is a great circle.

Solution : The equation

$x^2+y^2+z^2+7y-2z+2+\lambda(2x+3y+4z-8)=0$ represents a sphere passing through the circle given by

$$x^2+y^2+z^2+7y-2z+2=0$$

and $$2x+3y+4z-8=0$$

The centre of this sphere is $\left(-\lambda,-\frac{(7+3\lambda)}{2},1-2\lambda\right)$.

This should satisfy $2x+3y+4z-8=0$, as its centre should coincide with the centre of the sphere

$\therefore$ $$-2\lambda-\frac{21+9\lambda}{2}+4-8\lambda=8$$

or, $-4\lambda-21-9\lambda+8-16\lambda=16$ or $\lambda=-1$

$\therefore$ The required equation of the sphere is

$$x^2+y^2+z^2+7y-2z+2-2x-3y-4z+8=0$$

or, $$x^2+y^2+z^2-2x+4y-6z+10=0$$

CHECK YOUR PROGRESS 36.4

1. Find the centre and radius of each of the following circles :

   (i) $x^2+y^2+z^2-2y-4z-11=0$ and $x+2y+2z=15$

   (ii) $(x-3)^2+(y+2)^2+(z-1)^2=100$ and $2x-2y-z+9=0$

2. Show that the circle in which the sphere

$x^2 + y^2 + z^2 - 2x - 4y - 6z = 2$ is cut by the plane $x + 2y + 2z = 20$ has its centre at the point (2, 4, 5) with a radius of $\sqrt{3}$ units.

LET US SUM UP

- A sphere is the set of points in space such that its distance from a fixed point always remains constant. The fixed point in the space is called the centre and the constant distance is called the radius of the sphere.
- The equation of a sphere with centre (x_1, y_1, z_1) and radius r is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$$

This is called ***centre radius*** form of the equation of a sphere.

- The equation of a sphere with centre at origin and radius r is

$$x^2 + y^2 + z^2 = r^2$$

- A general equation of second degree in x, y and z represents a sphere if

(i) The co-efficients of the terms involving x^2, y^2 and z^2 are all equal.

(ii) There are no terms involving xy, yz or zx.

- The centre and radius of a sphere

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0 \text{ are } (-g, -f, -h)$$

and $\sqrt{g^2 + f^2 + h^2 - c}$ respectively.

- The equation of a sphere through four non-coplanar points

(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

- The equation of a sphere, with (x_1, y_1, z_1), (x_2, y_2, z_2) as extremities of a diameter, is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$$

- The plane section of a sphere with origin as centre and r, the radius is determined by joint equations :

$$x^2 + y^2 + z^2 = r^2 \text{ and } a(x - a) + b(y - b) + c(z - c) = 0$$

- The sphere $S \equiv x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0$ and the plane $L \equiv lx + my + nz + k = 0$, together represent the equation of a circle being intersection of S = 0 and L = 0.

  The equation $S + \lambda L = 0$ represents the equation of the sphere passing through the circle given by S = 0 and L = 0 together.

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. Find the equation of the sphere which passes through the point $(2, -2, 3)$ and which has its centre at (0, 0, 0).
2. Find the co-ordinates of the centre and the radius of the sphere

$$2(x^2 + y^2 + z^2) - 4x + 6y - 5z = 0$$

3. Find the equation of the sphere concentric with

   $x^2 + y^2 + z^2 - 2x - 4y - 6z - 11 = 0$ and with radius 10.
4. Find the locus of a point which moves such that the sum of the squares of its distances from the points (1, 2, 3), (2, –3, 5) and (0, 7, 4) is 147.
5. For the sphere $x^2 + y^2 + z^2 + 2x + 4y - 6z - 5 = 0$, find if the point $(1, -1, 3)$ lies in the interior or exterior of the sphere.
6. Find the equation of the sphere passing through the points (0, 0, 0), (1, 0, 0), (0, 2, 0) and (0, 0,3).
7. Find the equation of the sphere having extremities of one of its diameter as $(-1, 2, -3)$ and $(3, 1, -1)$.
8. If one end of a diameter of the sphere $x^2 + y^2 + z^2 - 7x - 3y + 1 = 0$ is the point (4, 5, 1), find the other end point of the diameter.
9. Find the equation of the sphere passing through the origin and cutting intercepts a, b and c from the positive directions of the co-ordinate axes.
10. Find the radius of the circular section of the sphere $x^2 + y^2 + z^2 = 49$ by the plane $2x + 3y - z - 5\sqrt{14} = 0$.
11. Find the equation of a sphere for which the circle $x^2 + y^2 + z^2 = 4$, $x + y + 4 = 0$ is a great circle.

Notes

ANSWERS

CHECK YOUR PROGRESS 36.1

1. $x^2 + y^2 + z^2 = 25$

2. Centre : $\left(\frac{1}{2}, -1, \frac{3}{2}\right)$; radius $= \sqrt{\frac{55}{6}}$

3. (i) $(x-3)^2 + (y+3)^2 + (z-1)^2 = 19$

   (ii) $(x-2)^2 + (y+2)^2 + (z-1)^2 = 9$

4. $(x-3)^2 + (y+3)^2 + (z-1)^2 = 9$

5. (i) interior (ii) on the sphere (iii) exterior

CHECK YOUR PROGRESS 36.2

1. (i) $x^2 + y^2 + z^2 - ax - by - cz = 0$, $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$; $\frac{\sqrt{a^2+b^2+c^2}}{2}$

   (ii) $\frac{x^2+y^2+z^2}{a^2+b^2+c^2} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0$;

   $\left(\frac{a^2+b^2+c^2}{2a}, \frac{a^2+b^2+c^2}{2b}; \frac{a^2+b^2+c^2}{2c}\right)$;

   $\frac{a^2+b^2+c^2}{2\,abc}\sqrt{b^2c^2+c^2a^2+a^2b^2}$

   (iii) $6\left(x^2+y^2+z^2\right) + 14x + 2y - 34z = 0$; $\left(-\frac{7}{6}, -\frac{1}{6}, \frac{17}{6}\right)$; $\frac{\sqrt{339}}{6}$

2. $$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 3 & 1 & -1 & -1 & 1 \\ 19 & 3 & 3 & 1 & 1 \\ 29 & -2 & 0 & 5 & 1 \\ 33 & -1 & 4 & 4 & 1 \end{vmatrix} = 0$$

CHECK YOUR PROGRESS 36.3

1. (i) $x^2 + y^2 + z^2 + 3x - 3y + 3z - 56 = 0$; $\left(-\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right)$; $\frac{\sqrt{251}}{2}$

(ii) $x^2 + y^2 + z^2 - x + 3y - 9z + 18 = 0$; $\left(\frac{1}{2}, -\frac{3}{2}, \frac{9}{2}\right)$; $\frac{\sqrt{19}}{2}$

(iii) $x^2 + y^2 + z^2 - 4x - 6y - 2z = 0$; (2, 3, 1) ; $\sqrt{14}$

2. $(3, -6, 2)$ 3. $\left(-\frac{3}{2}, 4, 2\right)$; $\frac{\sqrt{105}}{2}$

CHECK YOUR PROGRESS 36.4

1. (i) (1, 3, 4) ; $\sqrt{7}$ (ii) $(-1, 2, 3)$; 8

TERMINAL EXERCISE

1. $x^2 + y^2 + z^2 = 17$ 2. $\left(1, -\frac{3}{2}, \frac{5}{4}\right)$; $\frac{\sqrt{77}}{4}$
3. $x^2 + y^2 + z^2 - 2x - 4y - 6z - 86 = 0$
4. $x^2 + y^2 + z^2 - 2x - 4y - 8z - 10 = 0$
5. Interior of the sphere
6. $x^2 + y^2 + z^2 - x - 2y - 3z = 0$
7. $x^2 + y^2 + z^2 - 2x - 3y + 4z - 2 = 0$
8. $(3, -2, -1)$ 9. $x^2 + y^2 + z^2 - ax - by - cz = 0$
10. $2\sqrt{6}$ 11. $x^2 + y^2 + z^2 + 4x + 4y - 12 = 0$.