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APPLICATIONS OF DERIVATIVES

RATE OF CHANGE OF QUANTITIES

The value of $\frac{dy}{dx}$ $at\ x = x_0\ \ i.e\ \left(\frac{dy}{dx}\right)_{x=x_0} = f'(x_0)$

APPROXIMATIONS

$$\triangle y = \frac{dy}{dx}\Delta x$$

Absolute Error : The error Δ x in x is called the absolute error in x

RELATIVE ERROR : If Δx is an error in c then $\frac{\Delta x}{x}$ is called relative error in x

PERCENTAGE ERROR: If Δx is an error in x, then $\frac{\Delta x}{x}$ x 100 is called percentage error in x.

Slope of Tangent and

Normal

The equation of normal at (x_1, y_1) to the curve
$y = f(x)$ is

$$(y - y_1) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}(x - x_1)$$

or $$(y-y_1) . \left(\frac{dy}{dx}\right)_{(x_1,y_1)} + (x - x_1) = 0$$

EQUATIONS OF TANGENT AND NORMAL TO A CURVE

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}[x - x_1]$$

the equation of normal to the curve y = f(x) at the point (x_1,y_1) is

$$y - y_1 = \left(\frac{1}{\frac{dy}{dx}}\right)_{(x_1,y_1)}[x - x_1]$$

The equation of tangent to a curve is parallel to x-axis if

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0.$$

. In that case the equation of tangent is $y = y_1$

In case$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} \to$

∞, *tangent at* (x_1, y_1) *is parellel to y axis and its equation is* $x = x_1$

Rolle's Theorem

If a function f defined on the closed interval [a, b], is

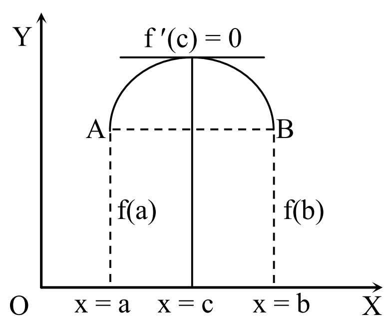
(i) Continuous on [a, b],

(ii) Derivable on (a, b) and

(iii) f(a) = f(b), then there exists atleast one real number c between a and b ($a < c < b$) such that $f'(c) = 0$

Geometrical interpretation

Let the curve y = f(x), which is continuous on [a, b] and derivable on (a, b), be drawn.



The theorem states that between two points with equal ordinates on the graph of f, there exists atleast one point where the tangent is parallel to x-axis.

Langrange's Mean Value Theorem

If a function f defined on the closed interval [a, b], is

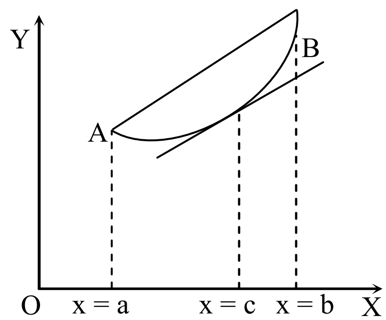
(i) Continuous on [a, b] and

(ii) Derivable on (a, b), then there exists atleast one real number c between a and b ($a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical interpretation

The theorem states that between two points A and B on the graph of f there exists atleast one point where the tangent is parallel to the chord AB.



INCREASING AND DECREASING FUNCTIONS

A function is said to be an increasing function in an interval if $f(x+h) > f(x)$ for all x belonging to the interval when h is positive.)

A function f(x) defined over the closed interval [a, b] is said to be a decreasing function in the given interval, if $f(x_2) \le f(x_1)$, whenever $x_2 > x_1$, $x_1, x_2 \in [a,b]$. It is saidto be strictly decreasing if $f(x_1) > f(x_2)$ for all $x_2 > x_1$, $x_1, x_2 \in [a,b]$

MONOTONIC FUNCTIONS

Monotonic Increasing :

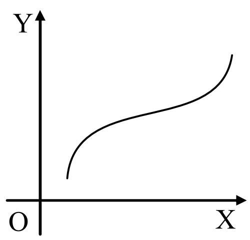
A function f(x) defined in a domain D is said to be monotonic increasing function if the value of f(x) does not decrease (increase) by increasing (decreasing) the value of x or

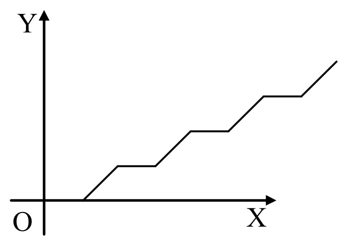
If

$$\begin{cases} x_1 < x_2 \Rightarrow f(x_1) \le f(x_2) \\ \text{or } x_1 < x_2 \Rightarrow f(x_1) \not> f(x_2), \forall x_1, x_2 \in D \end{cases}$$

or

$$\begin{cases} x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2) \\ \text{or } x_1 > x_2 \Rightarrow f(x_1) \not< f(x_2), \forall x_1, x_2 \in D \end{cases}$$





Monotonic Decreasing :

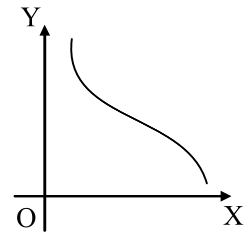
A function f(x) defined in a domain D is said to be monotonic decreasing function if the value of f(x) does not increase (decrease) by increasing (decreasing) the value of x or

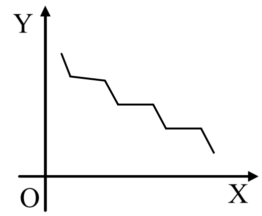
If

$$\begin{cases} x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2) \\ \text{or } x_1 < x_2 \Rightarrow f(x_1) \not< f(x_2), \forall x_1, x_2 \in D \end{cases}$$

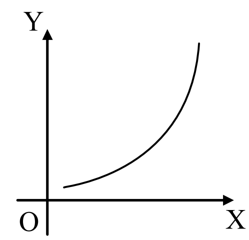
or

$$\begin{cases} x_1 > x_2 \Rightarrow f(x_1) \le f(x_2) \\ \text{or } x_1 > x_2 \Rightarrow f(x_1) \not> f(x_2), \forall x_1, x_2 \in D \end{cases}$$

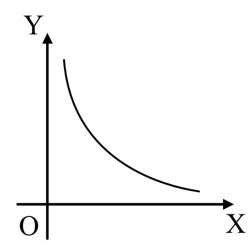




A function is said to be monotonic function in a domain if it is either monotonic increasing or monotonic decreasing in that domain.



Similarly if $x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)$, $\forall\, x_1, x_2 \in D$ then it is called strictly decreasing in domain D.



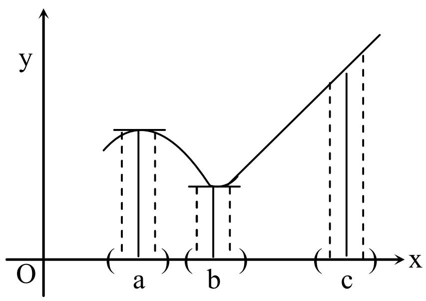
RELATION BETWEEN THE SIGN OF THE DERIVATIVE AND MONOTONICITY OF FUNCTION

MAXIMUM AND MINIMUM VALUES OF A FUNCTION

The value of a function f (x) is said to be maximum at x = a, if there exists a very small positive number h, such that

$f(x) < f(a)\ \forall\ x \in (a - h, a + h)$, $x \neq a$

In this case the point $x = a$ is called a point of maxima for the function $f(x)$.



Similarly, the value of $f(x)$ is said to the minimum

at $x = b$, If there exists a very small positive number, h, such that

$$f(x) > f(b), \forall x \in (b-h, b+h), x \neq b$$

In this case $x = b$ is called the point of minima for the function $f(x)$.

Hene we find that,

(i) $x = a$ is a maximum point of $f(x)$

$$\begin{cases} f(a) - f(a+h) > 0 \\ f(a) - f(a-h) > 0 \end{cases}$$

(ii) $x = b$ is a minimum point of $f(x)$

$$\begin{cases} f(b) - f(b+h) < 0 \\ f(b) - f(b-h) < 0 \end{cases}$$

(iii) $x = c$ is neither a maximum point nor a minimum point

$$\left\{ \begin{matrix} f(c) - f(c+h) \\ \text{and} \\ f(c) - f(c-h) > 0 \end{matrix} \right\}$$ have opposite signs.

A. Necessary Condition : A point $x = a$ is an extreme point of a function $f(x)$ if $f'(a) = 0$, provided $f'(a)$ exists. Thus if $f'(a)$ exists, then

$x = a$ is an extreme point $\Rightarrow f'(a) = 0$ or $f'(a) \neq 0 \Rightarrow x = a$ is not an extreme point.

But its converse is not true i.e.

$f'(a) = 0$ $x = a$ is an extreme point.

B. Sufficient Condition :

(i) The value of the function $f(x)$ at $x = a$ is maximum, if $f'(a) = 0$ and $f''(a) < 0$.

(ii) The value of the function $f(x)$ at $x = a$ in minimum if $f'(a) = 0$ and $f''(a) > 0 > 0$.

Check Yourself

1. When $x < 0$, function $f(x) = x^2$ is -
   (A) decreasing
   (B) increasing
   (C) constant
   (D) not monotonic
2. Function $f(x) = 2x^3 - 9x^2 + 12x + 29$ is decreasing when –
   (A) (A) $x < 2$ (B) $x > 2$
   (B) (C) $x > 3$ (D) $1 < x < 2$

3. The function $f(x) = \frac{|x|}{x}$ $(x \neq 0)$, $x > 0$ is -
   (A) (A) decreasing (B) increasing
   (B) (C) constant function (D) None of these

4. When $x \in (0, 1)$, function $f(x) = \frac{1}{\sqrt{x}}$ is
   (A) increasing
   (B) decreasing
   (C) neither increasing nor decreasing
   (D) constant

5. Function $f(x) = 3x^4 + 7x^2 + 3$ is
   (A) monotonically increasing
   (B) monotonically decreasing
   (C) not monotonic
   (D) odd function

6. For what values of x, the function $f(x) = x + \frac{4}{x^2}$ is monotonically decreasing
   (A) $x < 0$ (B) $x > 2$
   (A) (C) $x < 2$ (D) $0 < x < 2$

7. f(c) is a maximum value of f(x) if -
   (A) $f'(c) = 0$, $f''(c) > 0$
   (B) $f'(c) = 0$, $f''(c) < 0$
   (C) $f'(c) \neq 0$, $f''(c) = 0$
   (D) $f'(c) < 0$, $f''(c) > 0$

8. f(c) is a minimum value of f(x) if -
   (A) $f'(c) = 0$, $f''(c) > 0$
   (B) $f'(c) = 0$, $f''(c) < 0$
   (C) $f'(c) \neq 0$, $f''(c) = 0$
   (D) $f'(c) < 0$, $f''(c) > 0$

9. f(c) is a maximum value of f(x) when at x = c -
   (A) $f'(x)$ changes sign from +ve to –ve
   (B) $f'(x)$ changes sign from –ve to +ve
   (C) $f'(x)$ does not change sign
   (D) $f'(x)$ is zero

10. f(c) is a minimum value of f(x) when at x = c-
    (A) $f'(x)$ changes sign +ve to –ve
    (B) $f'(x)$ changes sign from –ve to +ve
    (C) $f'(x)$ does not change sign
    (D) $f'(x)$ is zero

Stretch Yourself

1. Find the maximum value of $\sin^3 x + \cos^3 x$
2. Let $f(x) = (x-1)^m (x-2)^n$ $(m, n \in N)$, $x \in R$. Then find point where

f(x) is either local maximum or local minimum

3. For the curve $\frac{c^4}{r^2} = \frac{a^2}{\sin^2\theta} + \frac{b^2}{\cos^2\theta}$, find the
   a. maximum value of r
4. Find the minimum and maximum value of
   a. $f(x, y) = 7x^2 + 4xy + 3y^2$ subjected to $x^2 + y^2 = 1$.

5. Let the function f(x) be defined as below,
6. $f(x) = \begin{cases} \sin^{-1}\lambda + x^2, 0 < x < 1 \\ 2x, \quad x \geq 1 \end{cases}$
   a. f(x) can have a minimum at x = 1 then find value of λ is -
7. If $a^2x^4 + b^2y^4 = c^6$, then find the maximum value of xy

Hint to Check Yourself

1 A 2 D 3 C 4 B 5 C

6 D 7 B 8 A 9 A 10 B