311en35

35

PLANE

Look closely at a room in your house. It has four walls, a roof and a floor. The floor and roof are parts of two parallel planes extending infinitely beyond the boundary. You will also see two pairs of parallel walls which are also parts of parallel planes. Similarly, the tops of tables, doors of rooms etc. are examples of parts of planes.

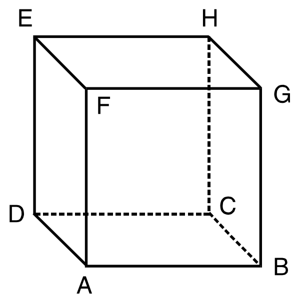

Fig. 35.1

If we consider any two points in a plane, the line joining these points will lie entirely in the same plane. This is the characteristic of a plane.

Look at Fig.35.1.You know that it is a representation of a rectangular box. This has six faces, eight vertices and twelve edges.

The pairs of opposite and parallel faces are

(i) ABCD and FGHE

(ii) AFED and BGHC

(iii) ABGF and DCHE

and the sets of parallel edges are given below :

(i) AB, DC EH and FG

(ii) AD, BC, GH and FE

(iii) AF, BG, CH and DE

Each of the six faces given above forms a part of the plane, and there are three pairs of parallel planes, denoted by the opposite faces.

In this lesson, we shall establish the general equation of a plane, the equation of a plane passing through three given points, the intercept form of the equation of a plane and the normal form of the equation of a plane. We shall show that a homogeneous equation of second degree in three variables x,y and z represents a pair of planes. We shall also find the equation of a plane bisecting the angle between two planes and area of a triangle in space.

OBJECTIVES

After studying this lesson, you will be able to :

- identify a plane;

- establish the equation of a plane in normal form;
- find the general equation of a plane passing through a given point;
- find the equation of a plane passing through three given points;
- find the equation of a plane in the intercept form and normal form;
- find the angle between two given planes;

EXPECTED BACKGROUND KNOWLEDGE

- Basic knowledge of three dimensional geometry.
- Direction cosines and direction ratio of a line.
- Projection of a line segment on another line.
- Condition of perpendicularity and parallelism of two lines in space.

35.1 VECTOR EQUATION OF A PLANE

A plane is uniquely determined if any one of the following is known:

(i) Normal to the plane and its distance from the origin is given.

(ii) One point on the plane is given and normal to the plane is also given.

(iii) It passes through three given non collinear points.

35.2 EQUATION OF PLANE IN NORMAL FROM

Let the distance (OA) of the plane from origin O be d and let $\hat{n}$ be a unit vector normal to the plane. Consider $\vec{r}$ as position vector of an arbitarary point P on the plane.

Since OA is the perpendicular distance of the plane from the origin and $\hat{n}$ is a unit vector perpendicular to the plane.

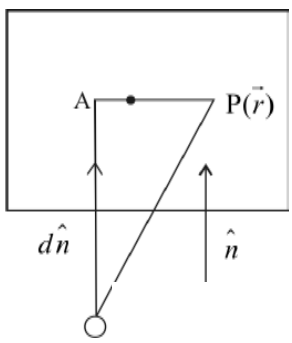

Fig. 35.2

$$\therefore \qquad \overrightarrow{OA} = d\hat{n}$$

$$\text{Now} \qquad \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - d\hat{n}$$

$\overrightarrow{OA}$ is perpendicular to the plane and $\overrightarrow{AP}$ lies in the plane, therefore $\overrightarrow{OA} \perp \overrightarrow{AP}$

$$\Rightarrow \qquad \overrightarrow{AP} . \overrightarrow{OA} = 0$$

$$\text{i.e.} \qquad (\vec{r} - d\hat{n}).\hat{n} = 0$$

$$\text{i.e.} \qquad \vec{r}.\hat{n} - d = 0$$

$$\text{i.e.} \qquad \vec{r}.\hat{n} = d \qquad \ldots(3)$$

which is the equation of plane in vector from.

35.3 CONVERSION OF VECTOR FORM INTO CARTESIAN FORM

Let (x, y, z) be the co-ordinates of the point P and l, m, n be the direction cosines of $\hat{n}$.

Then $\vec{r} \;=\; x\hat{i} + y\hat{j} + z\hat{k}$

$$\hat{n} \;=\; l\hat{i} + m\hat{j} + n\hat{k}$$

Substituting these value in equation (3) we get

$$(x\hat{i} + y\hat{j} + z\hat{k}).(l\hat{i} + m\hat{y} + n\hat{k}) \;=\; d$$

$$\Rightarrow \qquad lx + my + nz \;=\; d$$

This is the corresponding Cartesian form of equation of plane in normal form.

Note : In equation (3), if $\vec{r}.\vec{n} = d$ is the equation of the plane then d is not the distance of the plane from origin. To find the distance of the plane from origin we have to convert $\vec{n}$ into $\hat{n}$ by dividing both sides by $|\vec{n}|$. Therefore $\dfrac{d}{|\vec{n}|}$ is the distance of the plane from the origin.

Example 15.1 Find the distance of the plane $\vec{r}.(6\hat{i} - 3\hat{j} - 2\hat{k}) - 1 = 0$ from the origin. Also find the direction cosines of the unit vector perpendicular to the plane.

Solution : The given equation can be written as

$$\vec{r}.(6\hat{i} - 3\hat{j} - 2\hat{k}) \;=\; 1$$

$$|6\hat{i} - 3\hat{j} - 2\hat{k}| \;=\; \sqrt{36 + 9 + 4} = 7$$

Dividing both sides of given equation by 7 we get

$$\frac{\vec{r}.(6\hat{i} - 3\hat{j} - 2\hat{k})}{7} \;=\; \frac{1}{7}$$

i.e. $$\vec{r}.\left(\frac{6}{7}\hat{i} - \frac{3}{7}\hat{j} - \frac{2}{7}\hat{k}\right) \;=\; \frac{1}{7}$$

$\therefore$ d.c.'s of unit vector normal to the plane are $\dfrac{6}{7}, \dfrac{-3}{7}, \dfrac{-2}{7}$ and distance of plane from origin $= \dfrac{1}{7}$

35.4 EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND PERPENDICULAR TO A GIVEN VECTOR

Let $\vec{a}$ be the position vector of the given point A and $\vec{r}$ the position vector of an arbitrary point on the plane. $\vec{n}$ is a vector perpendicular to the plane.

Now $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= \vec{r} - \vec{a}$

Now $\vec{n} \perp (\vec{r} - \vec{a})$

$\therefore \quad (\vec{r} - \vec{a}) \cdot \vec{n} = 0 \quad ...(4)$

This is vector equation of plane in general form.

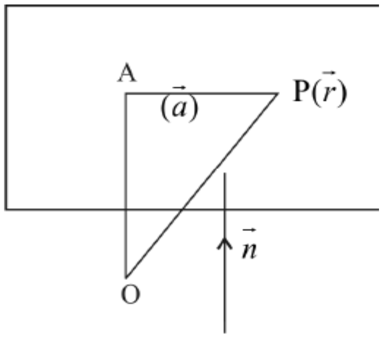

Fig. 35.3

35.5 CARTESIAN FORM

Let (x_1, y_1, z_1) be the coordinates of the given point A and (x, y, z) be the coordinates of point P. Again let a, b, c be the direction ratios of normal vector $\vec{n}$.

Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$

Substituting these values in equation (4) we get

$$\{(x - x_1)i + (y - y_1)j + (z - z_1)k\} \cdot \{ai + bj + ck\} = 0$$

$$\Rightarrow \quad a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

which is the corresponding Cartesian form of the equation of plane.

Example 35.2 Find the vector equation of a plane passing through the point (5, 5, –4) and perpendicular to the line with direction ratios 2, 3, –1.

Solution : Here $\vec{a} = 5\hat{i} + 5\hat{j} - 4\hat{k}$

and $\vec{n} = 2\hat{i} + 3\hat{j} - \hat{k}$

$\therefore$ Equation of plane is $\left(\vec{r} - (5\hat{i} + 5\hat{j} - 4\hat{k})\right) \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0$

35.6 EQUATION OF A PLANE PASSING THROUGH THREE NON COLLINEAR POINTS

(a) Vector Form

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of the given points Q, R and S respectively.

Let $\vec{r}$ be the position vector of an arbitrary point P on the plane.

Vectors $\overrightarrow{QR} = \vec{b} - \vec{a}, \overrightarrow{QS} = \vec{c} - \vec{a}$ and $\overrightarrow{QP} = \vec{r} - \vec{a}$ lie in the same plane and $\overrightarrow{QR} \times \overrightarrow{QS}$ is a vector perpendicular to both $\overrightarrow{QR}$ and $\overrightarrow{QS}$. Therefore $\overrightarrow{QR} \times \overrightarrow{QS}$ is perpendicular to $\overrightarrow{QP}$ also.

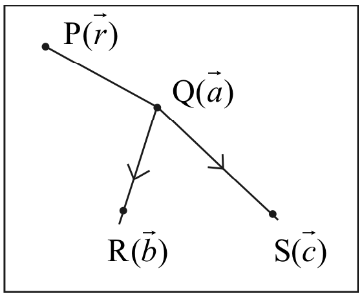

Fig. 35.4

$$\therefore \qquad \overrightarrow{QP} \cdot (\overrightarrow{QR} \times \overrightarrow{QS}) = 0$$

$$(\vec{r} - \vec{a}) \cdot \left\{(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right\} = 0 \qquad \ldots(5)$$

This is the equation of plane in vector form.

(b) Cartesian Form

Let (x, y, z), (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the coordinates of the points P, Q, R and S respectively.

$$\therefore \quad \overrightarrow{QP} = \vec{r} - \vec{a} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$$

$$\overrightarrow{QR} = \vec{b} - \vec{a} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\overrightarrow{QS} = \vec{c} - \vec{a} = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}$$

Substituting these values in equation (5) we get.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

which is the equation of plane in Cartesian form.

Example 35.3 Find the vector equation of the plane passing through the points Q(2, 5, –3), R(–2, –3, 5) and S(5, 3, –3).

Solution : Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of points Q, R and S respectively and $\vec{r}$ be the position vector of an arbitrary point on the plane.

Vector equation of plane is $\{\vec{r} - \vec{a}\} \cdot \left\{(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right\} = 0$

Here
$$\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$$
$$\vec{b} = -2\hat{i} - 3\hat{j} + 5\hat{k}$$
$$\vec{c} = 5\hat{i} + 3\hat{j} - 3\hat{k}$$

$$\vec{b}-\vec{a} = -4\hat{i}-8\hat{j}+8\hat{k}$$

$$\vec{c}-\vec{a} = 3\hat{i}-2\hat{j}$$

$\therefore$ Required equation is $\{\vec{r}-(2\hat{i}+5\hat{j}-3\hat{k})\}.\{(-4\hat{i}-8\hat{j}+8\hat{k})\times(3\hat{i}-2\hat{j})\} = 0$

35.7 EQUATION OF A PLANE IN THE INTERCEPT FORM

Let a, b,c be the lengths of the intercepts made by the plane on the x,y and z axes respectively. It implies that the plane passes through the points (a,0,0), (0,b,0) and (0,0,c)

Putting $x_1 = a$ $\quad y_1 = 0$ $\quad z_1 = 0$

$x_2 = 0$ $\quad y_2 = b$ $\quad z_2 = 0$

and $x_3 = 0$ $\quad y_3 = 0$ $\quad z_3 = c$ in (A),

we get the required equation of the plane as

$$\begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

which on expanding gives $bcx + acy + abz - abc = 0$

or $$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1 \qquad .....(B)$$

Equation (B) is called the ***Intercept form*** of the equation of the plane.

Example 35.4 Find the equation of the plane passing through the points (0,2,3), (2,0,3) and (2,3,0).

Solution : Using (A), we can write the equation of the plane as

$$\begin{vmatrix} x-0 & y-2 & z-3 \\ 2-0 & 0-2 & 3-3 \\ 2-0 & 3-2 & 0-3 \end{vmatrix} = 0$$

or $$\begin{vmatrix} x & y-2 & z-3 \\ 2 & -2 & 0 \\ 2 & 1 & -3 \end{vmatrix} = 0$$

or $\quad x(6-0)-(y-2)(-6)+(z-3)(2+4) = 0$

or $\quad 6x+6(y-2)+6(z-3) = 0$

or $\quad x+y-2+z-3 = 0 \quad$ or $\quad x+y+z = 5$

Example 35.5 Show that the equation of the plane passing through the points (2,2,0), (2,0,2) and (4,3,1) is $x = y + z$.

Solution : Equation of the plane passing through the point (2,2,0) is

$$a(x-2)+b(y-2)+cz=0 \qquad .....(i)$$

$\because$ (i) passes through the point (2, 0, 2)

$\therefore \quad a(2-2)+b(0-2)+2c=0$

or $\quad c=b \qquad .....(ii)$

Again (i) passes through the point (4, 3, 1)

$\therefore \quad a(4-2)+b(3-2)+c=0$

or $\quad 2a+b+c=0 \qquad ....(iii)$

From (ii) and (iii), we get $2a+2b=0$ or $a=-b$

$\therefore$ (i) becomes

$$-b(x-2)+b(y-2)+bz=0$$

or $\quad -(x-2)+y-2+z=0$

or $\quad y+z-x=0$

or $\quad x=y+z$

Example 35.6 Reduce the equation of the plane $4x-5y+6z-60=0$ to the intercept form. Find its intercepts on the co-ordinate axes.

Solution : The equation of the plane is

$$4x-5y+6z-60=0 \quad \text{or} \quad 4x-5y+6z=60 \qquad ....(i)$$

The equation (i) can be written as $\dfrac{4x}{60}-\dfrac{5y}{60}+\dfrac{6z}{60}=1$ or $\dfrac{x}{15}+\dfrac{y}{(-12)}+\dfrac{z}{10}=1$

which is the interecept form of the equation of the plane and the intercepts on the co-ordinate axes are $15, -12$ and 10 respectively.

Example 35.7 Reduce each of the following equations of the plane to the normal form :

(i) $2x-3y+4z-5=0$ (ii) $2x+6y-3z+5=0$

Find the length of perpendicular from origin upon the plane in both the cases.

Solution : (i) The equation of the plane is $2x-3y+4z-5=0 \qquad ....(A)$

Dividing (A) by $\sqrt{2^2+(-3^2)+4^2}$ or , by $\sqrt{29}$

we get, $$\frac{2x}{\sqrt{29}}-\frac{3y}{\sqrt{29}}+\frac{4z}{\sqrt{29}}-\frac{5}{\sqrt{29}}=0$$

or $$\frac{2x}{\sqrt{29}}-\frac{3y}{\sqrt{29}}+\frac{4z}{\sqrt{29}}=\frac{5}{\sqrt{29}}$$

which is the equation of the plane in the normal form.

$\therefore$ Length of the perpendicular is $\dfrac{5}{\sqrt{29}}$

(ii) The equation of the plane is $2x+6y-3z+5=0 \qquad .....(B)$

Dividing (B) by $\sqrt{2^2 + 6^2 + (-3)^2}$

or by -7 we get, [refer to corollary 2]

$$-\frac{2x}{7} - \frac{6y}{7} + \frac{3z}{7} - \frac{5}{7} = 0 \quad \text{or} \quad -\frac{2x}{7} - \frac{6y}{7} + \frac{3z}{7} = \frac{5}{7}$$

which is the required equation of the plane in the normal form.

$\therefore$ Length of the perpendicular from the origin upon the plane is $\frac{5}{7}$

Example 35.8 The foot of the perpendicular drawn from the origin to the plane is (4, –2, –5). Find the equation of the plane.

Solution : Let P be the foot of perpendicular drawn from origin O to the plane.

Then P is the point $(4, -2, -5)$.

The equation of a plane through the point $P(4, -2, -5)$ is

$$a(x-4) + b(y+2) + c(z+5) = 0 \qquad .....(i)$$

Now OP $\perp$ plane and direction cosines of OP are proportional to

$$4-0, -2-0, -5-0$$

i.e. $\qquad 4, -2, -5$.

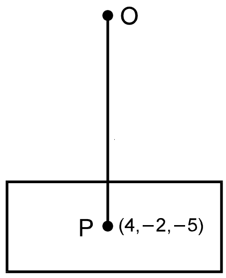

Fig. 35.5

Substituting $4, -2$ and -5 for a, b and c in (i), we get

$$4(x-4) - 2(y+2) - 5(z+5) = 0$$

or $\qquad 4x - 16 - 2y - 4 - 5z - 25 = 0$

or $\qquad 4x - 2y - 5z = 45$

which is the required equation of the plane.

CHECK YOUR PROGRESS 35.1

1. Reduce each of the following equations of the plane to the normal form :

   (i) $4x + 12y - 6z - 28 = 0$

   (ii) $3y + 4z + 3 = 0$

2. The foot of the perpendicular drawn from the origin to a plane is the point $(1, -3, 1)$. Find the equation of the plane.

3. The foot of the perpendicular drawn from the origin to a plane is the point (1, –2, 1). Find the equation of the plane.

4. Find the equation of the plane passing through the points

   (a) $\qquad (2, 2, -1), (3, 4, 2)$ and $(7, 0, 6)$

(b) $(2, 3, -3), (1, 1, -2)$ and $(-1, 1, 4)$

(c) $(2, 2, 2), (3, 1, 1)$ and $(6, -4, -6)$

5. Show that the equation of the plane passing through the points $(3, 3, 1)$, $(-3, 2 - 1)$ and $(8, 6, 3)$ is $4x + 2y - 13z = 5$

6. Find the equation of a plane whose intercepts on the coordinate axes are 2,3 and 4 respectively.

7. Find the intercepts made by the plane $2x + 3y + 4z = 24$ on the co-ordinate axes.

8. Show that the points $(-1, 4, -3)$, $(3, 2, -5)$, $(-3, 8, -5)$ and $(-3, 2, 1)$ are coplanar.

9. (i) What are the direction cosines of a normal to the plane $x - 4y + 3z = 7$.?

   (ii) What is the distance of the plane $2x + 3y - z = 17$ from the origin?

   (iii) The planes $\vec{r}.(\hat{i} - \hat{j} + 3\hat{k}) = 7$ and $\vec{r}.(3\hat{i} - 12\hat{j} - 5\hat{k}) = 6$ are ... to each other.

10. Convert the following equation of a plane in Cartesian form : $\vec{r}.(2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$.

11. Find the vector equation of a plane passing through the point (1, 1, 0), (1, 2, 1) and (–2, 2, –1).

12. Find the vector equation of a plane passing through the point (1, 4, 6) and normal to the vector $\hat{i} - 2\hat{j} + \hat{k}$.

35.6 ANGLE BETWEEN TWO PLANES

Let the two planes p_1 and p_2 be given by

$$a_1x + b_1y + c_1z + d_1 = 0 \qquad \text{....(i)}$$

and $$a_2x + b_2y + c_2z + d_2 = 0 \qquad \text{....(ii)}$$

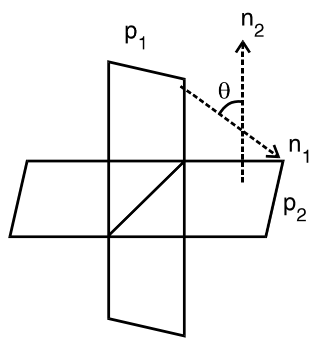

Fig. 35.5

Let the two planes intersect in the line l and let n_1 and n_2 be normals to the two planes. Let θ be the angle between two planes.

$\therefore$ The direction cosines of normals to the two planes are

$$\pm\frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \quad \pm\frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \quad \pm\frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$$

and $$\pm\frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \quad \pm\frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \quad \pm\frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}},$$

$\therefore$ $\cos\theta$ is given by $\cos\theta = \pm\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$

where the sign is so chosen that $\cos\ \theta$ is positive

Corollary 1 :

When the two planes are perpendicular to each other then $\theta = 90°$ i.e., $\cos\theta = 0$

$\therefore$ The condition for two planes $a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$ to be perpendicular to each other is

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Corollary 2 :

If the two planes are parallel, then the normals to the two planes are also parallel

$$\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$\therefore$ The condition of parallelism of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and

$$a_2x + b_2y + c_2z + d_2 = 0 \text{ is } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

This implies that the equations of two parallel planes differ only by a constant. Therefore, any plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is a constant.

Example 35.9 Find the angle between the planes

$$3x + 2y - 6z + 7 = 0 \qquad \text{......(i)}$$

and $$2x + 3y + 2z - 5 = 0 \qquad \text{......(ii)}$$

Solution : Here $a_1 = 3, b_1 = 2, c_1 = -6$

and $a_2 = 2, b_2 = 3, c_2 = 2$

$\therefore$ If θ is the angle between the planes (i) and (ii), then

$$\cos\theta = \frac{3.2 + 2.3 + (-6).2}{\sqrt{3^2 + 2^2 + (-6)^2}\ \sqrt{2^2 + 3^2 + 2^2}} = 0$$

$\therefore$ $\theta = 90°$

Thus the two planes given by (i) and (ii) are perpendicular to each other.

Example 35.10 Find the equation of the plane parallel to the plane $x - 3y + 4z - 1 = 0$ and passing through the point $(3, 1, -2)$.

Solution : Let the equation of the plane parallel to the plane

9. (i) What are the direction cosines of a normal to the plane $x - 4y + 3z = 7$.?
   (ii) What is the distance of the plane $2x + 3y - z = 17$ from the origin?

(iii) The planes $\vec{r}.(\hat{i}-\hat{j}+3\hat{k})=7$ and $\vec{r}.(3\hat{i}-12\hat{j}-5\hat{k})=6$ are ... to each other.

10. Convert the following equation of a plane in Cartesian form : $\vec{r}.(2\hat{i}+3\hat{j}-4\hat{k})=1$.

11. Find the vector equation of a plane passing through the point (1, 1, 0), (1, 2, 1) and (–2, 2, –1).

12. Find the vector equation of a plane passing through the point (1, 4, 6) and normal to the vector $\hat{i}-2\hat{j}+\hat{k}$.

$$x - 3y + 4z - 1 = 0 \text{ be } x - 3y + 4z + k = 0 \qquad \text{..... (i)}$$

Since (i) passes through the point $(3, 1, -2)$, it should satisfy it

$\therefore \qquad 3 - 3 - 8 + k = 0 \quad \text{or} \quad k = 8$

$\therefore$ The required equation of the plane is $x - 3y + 4z + 8 = 0$

Example 35.11 Find the equation of the plane passing through the points $(-1, 2, 3)$ and $(2, -3, 4)$ and which is perpendicular to the plane $3x + y - z + 5 = 0$

Solution : The equation of any plane passing through the point $(-1, 2, 3)$ is

$$a(x+1) + b(y-2) + c(z-3) = 0 \qquad \text{.....(i)}$$

Since the point $(2, -3, 4)$ lies on the plane (i)

$\therefore \qquad 3a - 5b + c = 0 \qquad \text{....(ii)}$

Again the plane (i) is perpendicular to the plane $3x + y - z + 5 = 0$

$\therefore \qquad 3a + b - c = 0 \qquad \text{....(iii)}$

From (ii) and (iii), by cross multiplication method, we get,

$$\frac{a}{4} = \frac{b}{6} = \frac{c}{18} \quad \text{or} \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{9}$$

Hence the required equation of the plane is

$$2(x+1) + 3(y-2) + 9(z-3) = 0 \quad \text{....[From (i)]}$$

or $\qquad 2x + 3y + 9z = 31$

Example 35.12 Find the equation of the plane passing through the point $(2, -1, 5)$ and perpendicular to each of the planes

$$x + 2y - z = 1 \qquad \text{and } 3x - 4y + z = 5$$

Solution : Equation of a plane passing through the point $(2, -1, 5)$ is

$$a(x-2) + b(y+1) + c(z-5) = 0 \qquad \text{.....(i)}$$

As this plane is perpendicular to each of the planes

$$x + 2y - z = 1 \qquad \text{and} \qquad 3x - 4y + z = 5$$

We have $\qquad a.1 + b.2 + c.(-1) = 0$

and $\qquad a.3 + b.(-4) + c.(1) = 0$

or $$a + 2b - c = 0 \qquad ....(ii)$$
$$3a - 4b + c = 0 \qquad ....(iii)$$

From (ii) and (iii), we get

$$\frac{a}{2-4} = \frac{b}{-3-1} = \frac{c}{-4-6}$$

or $$\frac{a}{-2} = \frac{b}{-4} = \frac{c}{-10} \quad \text{or} \quad \frac{a}{1} = \frac{b}{2} = \frac{c}{5} = \lambda \text{ (say)}$$

$\therefore$ $\quad a = \lambda,\ b = 2\lambda$ and $c = 5\lambda$

Substituting for a, b and c in (i), we get

$$\lambda(x-2) + 2\lambda(y+1) + 5\lambda(z-5) = 0$$

or $$x - 2 + 2y + 2 + 5z - 25 = 0$$

or $$x + 2y + 5z - 25 = 0$$

which is the required equation of the plane.

CHECK YOUR PROGRESS 35.2

1. Find the angle between the planes
   (i) $2x - y + z = 6$ and $x + y + 2z = 3$
   (ii) $3x - 2y + z + 17 = 0$ and $4x + 3y - 6z + 25 = 0$
2. Prove that the following planes are perpendicular to each other.
   (i) $x + 2y + 2z = 0$ and $2x + y - 2z = 0$
   (ii) $3x + 4y - 5z = 9$ and $2x + 6y + 6z = 7$
3. Find the equation of the plane passing through the point $(2, 3, -1)$ and parallel to the plane $2x + 3y + 6z + 7 = 0$
4. Find the equation of the plane through the points $(-1, 1, 1)$ and $(1, -1, 1)$ and perpendicular to the plane $x + 2y + 2z = 5$
5. Find the equation of the plane which passes through the origin and is perpendicular to each of the planes $x + 2y + 2z = 0$ and $2x + y - 2z = 0$

35.9 DISTANCE OF A POINT FROM A PLANE

Let the equation of the plane in normal form be

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p \text{ where } p>0 \qquad ....(i)$$

Case I : Let the point $P(x', y', z')$ lie on the same side of the plane in which the origin lies.

Let us draw a plane through point P parallel to plane (i).Its equation is

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p' \qquad .....(ii)$$

where p' is the length of the perpendicular drawn from origin upon the plane given by (ii). Hence the perpendicular distance of P from plane (i) is $p - p'$

As the plane (ii) passes through the point (x', y', z'),

$$x'\cos\alpha + y'\cos\beta + z'\cos\gamma = p'$$

$\therefore$ The distance of P from the given plane is

$$p - p' = p - (x'\cos\alpha + y'\cos\beta + z'\cos\gamma)$$

Case II : If the point P lies on the other side of the plane in which the origin lies, then the distance of P from the plane (i) is,

$$p' - p = x'\cos\alpha + y'\cos\beta + z'\cos\gamma - p$$

Note : If the equation of the plane be given as $ax + by + cz + d = 0$, we have to first convert it into the normal form, as discussed before, and then use the above formula.

Example 35.13 Find the distance of the point (1,2,3) from the plane $3x - 2y + 5z + 17 = 0$

Solution : Required distance $= \dfrac{3.1 - 2.2 + 5.3 + 17}{\sqrt{3^2 + (-2)^2 + 5^2}} = \dfrac{31}{\sqrt{38}}$ units.

Example 35.14 Find the distance between the planes

$$x - 2y + 3z - 6 = 0$$

and

$$2x - 4y + 6z + 17 = 0$$

Solution : The equations of the planes are

$$x - 2y + 3z - 6 = 0 \qquad \text{....(i)}$$

$$2x - 4y + 6z + 17 = 0 \qquad \text{....(ii)}$$

Here $\qquad \dfrac{1}{2} = \dfrac{(-2)}{(-4)} = \dfrac{3}{6}$

$\therefore$ Planes (i) and (ii) are parallel

Any point on plane (i) is (6, 0, 0)

$\therefore$ Distance between planes (i) and (ii) = Distance of point (6,0,0) from (ii)

$$= \frac{2 \times 6 - 4.0 + 6.0 + 17}{\sqrt{(2)^2 + (-4)^2 + 6^2}}$$

$$= \frac{29}{\sqrt{56}} \text{ units } = \frac{29}{2\sqrt{14}} \text{ units}$$

CHECK YOUR PROGRESS 35.3

1. Find the distance of the point

   (i) $(2, -3, 1)$ from the plane $5x - 2y + 3z + 11 = 0$

   (ii) $(3, 4, -5)$ from the plane $2x - 3y + 3z + 27 = 0$

2. Find the distance between planes

$3x + y - z - 7 = 0$ and $6x + 2y - 2z + 11 = 0$

LET US SUM UP

- A plane is a surface such that if any two points are taken on it, the line joining these two points lies wholly in the plane.
- $\vec{r}.\hat{n} = d$ is the vector equation of a plane where $\hat{n}$ is a unit vector normal to the plane and d is the distance of the plane from origin.
- Corresponding cartesian form of the equation is $lx + my + nz = d$, where l,m,n are the direction cosines of the normal vector to the plane and d is the distance of the plane from origin.
- $\left(\vec{r} - \vec{a}\right).\vec{n} = 0$ is another vecter equation of a plane where $\vec{a}$ is position vecter of a given point on the plane and $\vec{n}$ is a vecter normal to the plane.
- Corresponding cartesion form of this equation is $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$, where a,b,c are the direction ratios of normal to the plane and (x_1,y_1,z_1) are coordinates of given point on plane.
- $\left(\vec{r} - \vec{a}\right).\{\left(\vec{b} - \vec{a}\right) \times \left(\vec{c} - \vec{a}\right)\} = 0$ is the equation of a plane possing through three points with position vecter $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.
- Its corresponding cartesian equation is:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- Equation of a plane in the intercept from is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

  where a,b and c are intercepts made by the plane on x,y and z axes respectively.
- Angle θ between two planes $a_1x + b_1y + c_1z + d_1 = 0$

  and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

- Two planes are perpendicular to each other if and only if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

- Two planes are parallel if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- Distance of a point (x', y', z') from a plane

  $$x \cos\alpha + y \cos\beta + z \cos\gamma = p \text{ is}$$

  $|p - (x'\cos\alpha + y'\cos\beta + z'\cos\gamma)|$, where the point (x', y', z') lies on the same side of the plane in which the origin lies.

SUPPORTIVE WEB SITES

http://www.mathopenref.com/plane.html

http://en.wikipedia.org/wiki/Plane_(geometry)

TERMINAL EXERCISE

1. Find the equation of a plane passing through the point $(-2, 5, 4)$
2. Find the equation of a plane which divides the line segment joining the points $(2, 1, 4)$ and $(2, 6, 4)$ internally in the ratio of $2 : 3$.
3. Find the equation of the plane through the points $(1, 1, 0)$, $(1, 2, 1)$ and $(-2, 2, -1)$.
4. Show that the four points $(0, -1, -1)$, $(4, 5, 1)$, $(3, 9, 4)$ and $(-4, 4, 4)$ are coplanar. Also find the equation of the palne in which they lie.
5. The foot of the perpendicular drawn from $(1, -2, -3)$ to a plane is $(3, 2, -1)$. Find the equation of the plane.
6. Find the angle between the planes $x + y + 2z = 9$ and $2x - y + z = 15$
7. Prove that the planes $3x - 5y + 8z - 2 = 0$ and $12x - 20y + 32z + 9 = 0$ are parallel.
8. Determine the value of k for which the planes $3x - 2y + kz - 1 = 0$ and $x + ky + 5z + 2 = 0$ may be perpendicular to each other.
9. Find the distance of the point $(3, 2, -5)$ from the plane $2x - 3y - 5z = 7$
10. Find the vector equation of a plane possing through the point (3,-1,5) and perpendicular to the line with direction ratios (2,-3,1).
11. Find the vector equation of a plane perpendicular to the vecter $3\hat{i} + 5\hat{j} - 6\hat{k}$ and at a distance of 7 units from origin.
12. Find the vector equation of a plane passing through the points A(-2,6,-6), B(-3,10,-9), and C(-5,0,-6).

ANSWERS

CHECK YOUR PROGRESS 35.1

1. (i) $\frac{4x}{14}+\frac{12y}{14}-\frac{6z}{14}=2$ (ii) $-\frac{3}{5}y-\frac{4}{5}z=\frac{3}{5}$

2. $x-3y+z-11=0$ 3. $x-2y+z-6=0$ 4. (a) $5x+2y-3z-17=0$

(b) $3x-y+2=0$ (c) $x+2y-2=4$ 6. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$

7. Intercepts on x,y & z axes are 12,8,6 respectively.

9. (i) $\frac{1}{\sqrt{26}},\frac{-4}{\sqrt{26}},\frac{3}{\sqrt{26}}$ (ii) $\frac{17}{\sqrt{14}}$ units (iii) perpendicular

10. $2x+3y-4z=1$ 11. $\vec{x}.\left(2\hat{x}+3\hat{y}-3\hat{k}\right)=5$ 12. $\vec{x}.\left(\hat{x}-2\hat{y}+\hat{k}\right)+1=0$

CHECK YOUR PROGRESS 35.2

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{2}$ 3. $2x+3y+6z=7$ 4. $2x+2y-3z+3=0$

5. $2x-2y+z=0$

CHECK YOUR PROGRESS 35.3

1. (i) $\frac{30}{\sqrt{38}}$ units (ii) $\frac{6}{\sqrt{22}}$ units. 2. $\frac{25}{2\sqrt{11}}$ units.

TERMINAL EXERCISE

1. $a(x+2)+b(y-5)+c(z-4)=0$

2. $a(x-2)+b(y-3)+c(z-4)=0$

3. $2x+3y-3z-5=0$ 4. $5x-7y+11z+4=0$ 5. $x+2y+z=6$

6. $\frac{\pi}{3}$ 8. $k=-1$ 9. $\frac{18}{\sqrt{38}}$ 10. $\{\vec{r}-\left(-3\hat{i}+\hat{j}+5\hat{k}\right)\}.\left(2\hat{i}-3\hat{j}+\hat{k}\right)=0$

11. $\vec{r}.\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}\}=7$

12. $\{\vec{x}.\left(-2\hat{i}+6\hat{j}-6\hat{k}\right)\}.\{\left(-\hat{i}+4\hat{j}-3\hat{k}\right)\times\left(-3\hat{i}-6\hat{j}\right)\}=0$