# 34

311en34

# VECTORS

In day to day life situations, we deal with physical quantities such as distance, speed, temperature, volume etc. These quantities are sufficient to describe change of position, rate of change of position, body temperature or temperature of a certain place and space occupied in a confined portion respectively. We also come across physical quantities such as dispacement, velocity, acceleration, momentum etc. which are of a difficult type.

Let us consider the following situation. Let A, B, C and D be four points equidistant (say 5 km each) from a fixed point P. If you are asked to travel 5 km from the fixed point P, you may reach either A, B, C, or D. Therefore, only starting (fixed point) and distance covered are not sufficient to describe the destination. We need to specify end point (terminal point) also. This idea of terminal point from the fixed point gives rise to the need for direction.

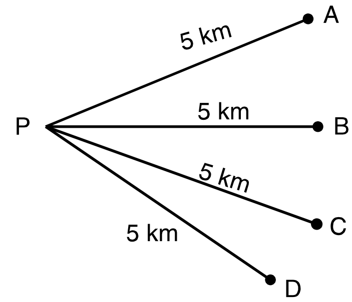



**Fig. 34.1**

Consider another example of a moving ball. If we wish to predict the position of the ball at any time what are the basics we must know to make such a prediction?

Let the ball be initially at a certain point A. If it were known that the ball travels in a straight line at a speed of 5cm/sec, can we predict its position after 3 seconds ? Obviously not. Perhaps we may conclude that the ball would be 15 cm away from the point A and therefore it will be at some point on the circle with A as its centre and radius 15 cms. So, the mere knowledge of speed and time taken are not sufficient to predict the position of the ball. However, if we know that the ball moves in a direction due east from A at a speed of 5cm/sec., then we shall be able to say that after 3 seconds, the ball must be precisely at the point P which is 15 cms in the direction east of A.

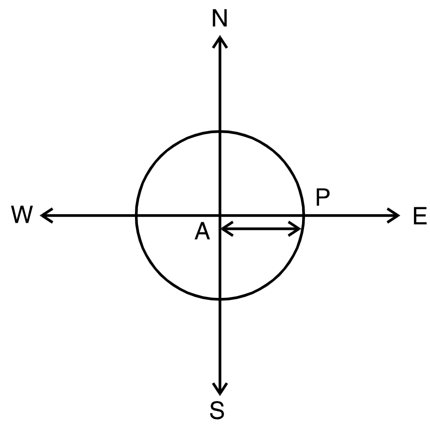



**Fig. 34.2**

Thus, to study the displacement of a ball after time t (3 seconds), we need to know the magnitude of its speed (i.e. 5 cm/sec) and also its direction (east of A)

In this lesson we will be dealing with quantities which have magnitude only, called scalars and the quantities which have both magnitude and direction, called vectors. We will represent vectors as directed line segments and determine their magnitudes and directions. We will study about various types of vectors and perform operations on vectors with properties thereof. We will also acquaint ourselves with position vector of a point w.r.t. some origin of reference. We will find out the resolved parts of a vector, in two and three dimensions, along two and three mutually perpendicular directions respectively. We will also derive section formula and apply that to problems. We will also define scalar and vector products of two vectors.

## OBJECTIVES

After studying this lesson, you will be able to :

- explain the need of mentioning direction;
- define a scalar and a vector;
- distinguish between scalar and vactor;
- represent vectors as directed line segment;
- determine the magnitude and direction of a vector;
- classify different types of vectors-null and unit vectors;
- define equality of two vectors;
- define the position vector of a point;
- add and subtract vectors;
- multiply a given vector by a scalar;
- state and use the properties of various operations on vectors;
- comprehend the three dimensional space;
- resolve a vector along two or three mutually prependicular axes;
- derive and use section formula; and
- define scalar (dot) and vector (cross) product of two vectors.
- define and understand direction cosines and direction ratios of a vector.
- define triple product of vectors.
- understand scalar triple product of vectors and apply it to find volume of a rectangular parallelopiped.
- understand coplanarity of four points.

## EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of plane and coordinate geometry.
- Knowledge of Trigonometry.

## 34.1 SCALARS AND VECTORS

A physical quantity which can be represented by a number only is known as a scalar i.e, quantities which have only magnitude. Time, mass, length, speed, temperature, volume, quantity of heat, work done etc. are all ***scalars***.

The physical quantities which have magnitude as well as direction are known as vectors. Displacement, velocity, acceleration, force, weight etc. are all examples of ***vectors***.

## 34.2 VECTOR AS A DIRECTED LINE SEGMENT

You may recall that a line segment is a portion of a given line with two end points. Take any line $l$ (called a support). The portion of L with end points A and B is called a line segment. The line segment AB along with direction from A to B is written as $\overrightarrow{AB}$ and is called a directed line segment. A and B are respectively called the initial point and terminal point of the vector $\overrightarrow{AB}$.

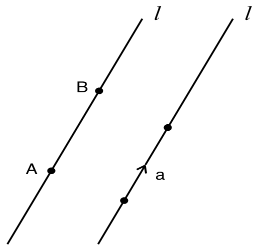



Fig. 34.3

The length AB is called the ***magnitude*** or ***modulus*** of $\overrightarrow{AB}$ and is denoted by $|\overrightarrow{AB}|$. In other words the length AB = $|\overrightarrow{AB}|$.

Scalars are usually represented by a, b, c etc. whereas vectors are usually denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$ etc. Magnitude of a vector $\vec{a}$ i.e., $|\vec{a}|$ is usually denoted by 'a'.

## 34.3 CLASSIFICATION OF VECTORS

### 34.3.1 Zero Vector (Null Vector)

A vector whose magnitude is zero is called a zero vector or ***null vector***. Zero vector has not definite direction. $\overrightarrow{AA}$, $\overrightarrow{BB}$ are zero vectors. Zero vectors is also denoted by $\vec{0}$ to distinguish it from the scalar 0.

### 34.3.2 Unit Vector

A vector whose magnitude is unity is called a ***unit vector***. So for a unit vector $\vec{a}$, $|\vec{a}| = 1$. A unit vector is usually denoted by $\hat{a}$. Thus, $\vec{a} = |\vec{a}|\,\hat{a}$.

### 34.3.3 Equal Vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude. i.e., $|\vec{a}| = |\vec{b}|$ and the same direction as shown in Fig. 14.4. Symbolically, it is denoted by $\vec{a} = \vec{b}$.

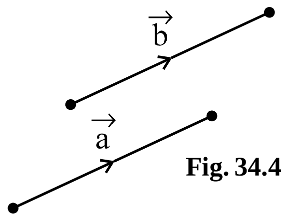



Fig. 34.4

***Remark :*** Two vectors may be equal even if they have different parallel lines of support.

### 34.3.4 Like Vectors

Vectors are said to be like if they have same direction whatever be their magnitudes. In the adjoining Fig. 14.5, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are like vectors, although their magnitudes are not same.

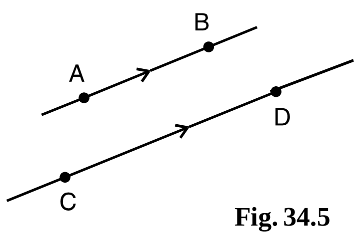



Fig. 34.5

### 34.3.5 Negative of a Vector

$\overrightarrow{BA}$ is called the ***negative of the vector*** $\overrightarrow{AB}$, when they have the same magnitude but opposite directions.

i.e. $\overrightarrow{BA} = -\overrightarrow{AB}$

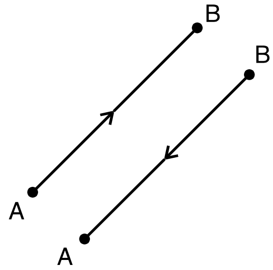



Fig. 34.6

### 34.3.6 Co-initial Vectors

Two or more vectors having the same initial point are called ***Co-initial vectors***.

In the adjoining figure, $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AC}$ are co-initial vectors with the same initial point A.

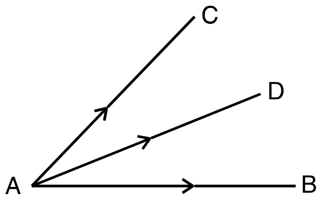



Fig. 34.7

### 34.3.7 Collinear Vectors

Vectors are said to be collinear when they are parallel to the same line whatever be their magnitudes. In the adjoining figure, $\overrightarrow{AB}$, $\overrightarrow{CD}$ and $\overrightarrow{EF}$ are collinear vectors. $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are also ***collinear***.

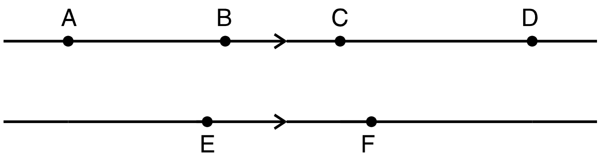



Fig. 34.8

### 34.3.8 Co-planar Vectors

Vectors are said to be co-planar when they are parallel to the same plane. In the adjoining figure $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are co-planar. Whereas $\vec{a}$, $\vec{b}$ and $\vec{c}$ lie on the same plane, $\vec{d}$ is parallel to the plane of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

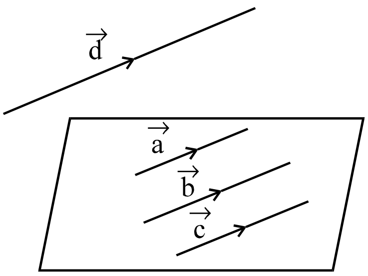



Fig. 34.9

***Note :*** (i) A zero vector can be made to be collinear with any vector.

(ii) Any two vectors are always co-planar.

**Example 34.1** State which of the following are scalars and which are vectors. Give reasons.

| | | | | | |
|---|---|---|---|---|---|
| (a) | Mass | (b) | Weight | (c) | Momentum |
| (d) | Temperature | (e) | Force | (f) | Density |

**Solution :** (a), (d) and (f) are scalars because these have only magnitude while (b), (c) and (e) are vectors as these have magnitude and direction as well.

**Example 34.2** Represent graphically

(a) a force 40N in a direction 60° north of east.

(b) a force of 30N in a direction 40° east of north.

**Solution :**

(a)

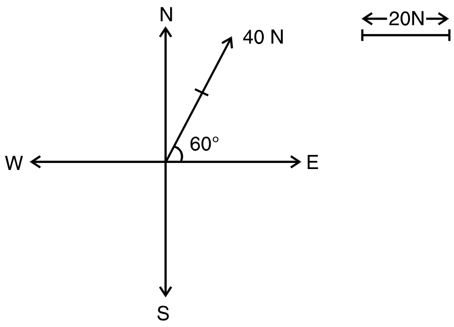



Fig. 34.10

( b)

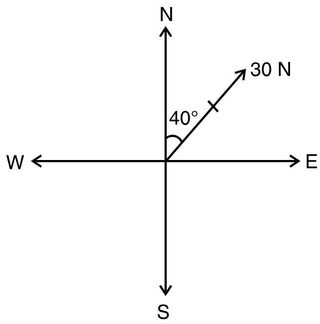



Fig. 34.11

## CHECK YOUR PROGRESS 34.1

1. Which of the following is a scalar quantity ?

   (a) Displacement (b) Velocity (c) Force (d) Length.

2. Which of the following is a vector quantity ?

   (a) Mass (b) force (c) time (d) tempertaure

3. You are given a displacement vector of 5 cm due east. Show by a diagram the corresponding negative vector.

4. Distinguish between like and equal vectors.

5. Represent graphically

   (a) a force 60 Newton is a direction 60° west of north.

   (b) a force 100 Newton in a direction 45° north of west.

## 34.4 ADDITION OF VECTORS

Recall that you have learnt four fundamental operations viz. addition, subtraction, multiplication and division on numbers. The addition (subtraction) of vectors is different from that of numbers (scalars).

In fact, there is the concept of resultant of two vectors (these could be two velocities, two forces etc.) We illustrate this with the help of the following example :

Let us take the case of a boat-man trying to cross a river in a boat and reach a place directly in the line of start. Even if he starts in a direction perpendicular to the bank, the water current carries him to a place different from the place he desired., which is an example of the effect of two velocities resulting in a third one called the resultant velocity.

Thus, two vectors with magnitudes 3 and 4 may not result, on addition, in a vector with magnitude 7. It will depend on the direction of the two vectors i.e., on the angle between them. The addition of vectors is done in accordance with the triangle law of addition of vectors.

### 34.4.1 Triangle Law of Addition of Vectors

A vector whose effect is equal to the resultant (or combined) effect of two vectors is defined as the resultant or sum of these vectors. This is done by the triangle law of addition of vectors.

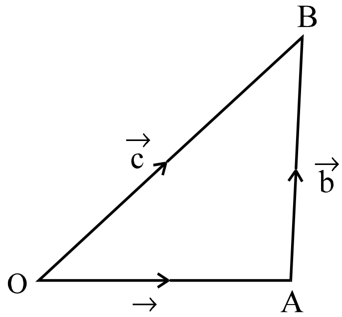


Fig. 34.12

In the adjoining Fig. 32.12 vector $\overrightarrow{OB}$ is the resultant or sum of vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ and is written as

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

i.e. $$\vec{a} + \vec{b} = \overrightarrow{OB} = \vec{c}$$

You may note that the terminal point of vector $\vec{a}$ is the initial point of vector $\vec{b}$ and the initial point of $\vec{a} + \vec{b}$ is the initial point of $\vec{a}$ and its terminal point is the terminal point of $\vec{b}$.

### 34.4.2 Addition of more than two Vectors

Addition of more then two vectors is shown in the adjoining figure

$$\vec{a} + \vec{b} + \vec{c} + \vec{d}$$
$$= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$$
$$= \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DE}$$
$$= \overrightarrow{AD} + \overrightarrow{DE}$$
$$= \overrightarrow{AE}$$

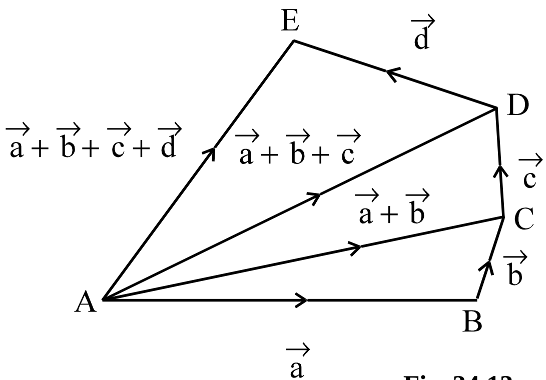


Fig. 34.13

The vector $\overrightarrow{AE}$ is called the sum or the resultant vector of the given vectors.

### 34.4.3 Parallelogram Law of Addition of Vectors

Recall that two vectors are equal when their magnitude and direction are the same. But they could be parallel [refer to Fig. 14.14].

See the parallelogram OABC in the adjoining figure :

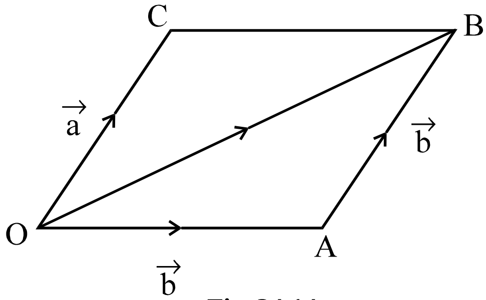


Fig. 34.14

We have, $$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

But $$\overrightarrow{AB} = \overrightarrow{OC}$$

$$\therefore \quad \overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB}$$

which is the parallelogram law of addition of vectors. **If two vectors are represented by the two adjacent sides of a parallelogram, then their resultant is represented by the diagonal through the common point of the adjacent sides.**

### 34.4.4 Negative of a Vector

For any vector $\vec{a} = \overrightarrow{OA}$, the negative of $\vec{a}$ is represented by $\overrightarrow{AO}$. The negative of $\overrightarrow{AO}$ is the same as $\overrightarrow{OA}$. Thus, $|\overrightarrow{OA}| = |\overrightarrow{AO}| = |\vec{a}|$ and $\overrightarrow{OA} = -\overrightarrow{AO}$. It follows from definition that for any vector $\vec{a}$, $\vec{a} + (-\vec{a}) = \vec{0}$.

### 34.4.5 The Difference of Two Given Vectors

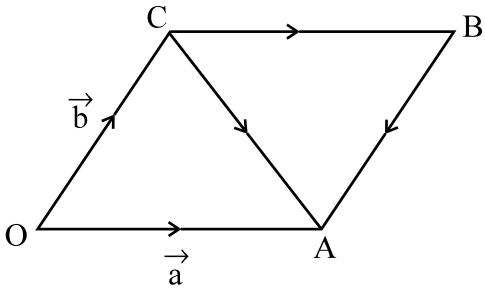


Fig. 34.15

For two given vectors $\vec{a}$ and $\vec{b}$, the difference $\vec{a} - \vec{b}$ is defined as the sum of $\vec{a}$ and the negative of the vector $\vec{b}$. i.e., $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

In the adjoining figure if $\overrightarrow{OA} = \vec{a}$ then, in the parallelogram OABC, $\overrightarrow{CB} = \vec{a}$

and $\quad \overrightarrow{BA} = -\vec{b}$

$\therefore \quad \overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA} = \vec{a} - \vec{b}$

**Example 34.3** When is the sum of two non-zero vectors zero ?

**Solution :** The sum of two non-zero vectors is zero when they have the same magnitude but opposite direction.

**Example 34.4** Show by a diagram $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

**Solution :** From the adjoining figure, resultant

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$= \vec{a} + \vec{b} \qquad \text{....(i)}$$

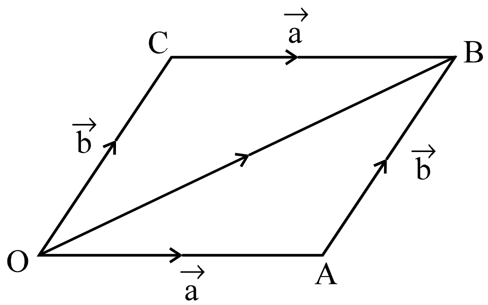


Fig. 34.16

Complete the parallelogram OABC

$$\overrightarrow{OC} = \overrightarrow{AB} = \vec{b},\ \overrightarrow{CB} = \overrightarrow{OA} = \vec{a}$$

$\therefore \quad \overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

$$= \vec{b} + \vec{a} \qquad \text{....(ii)}$$

$\therefore \quad \vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad$ [ From (i) and (ii) ]

## CHECK YOUR PROGRESS 34.2

1. The diagonals of the parallelogram ABCD intersect at the point O. Find the sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$.

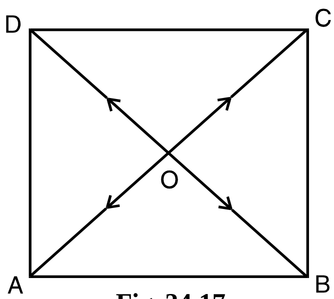


Fig. 34.17

MODULE - IX
Vectors and three dimensional Geometry

Notes

2. The medians of the triangle ABC intersect at the point O. Find the sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

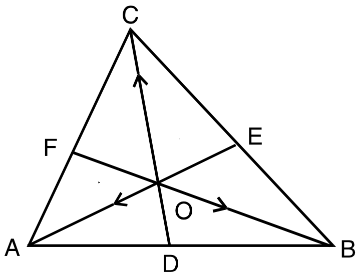


Fig. 34.18

## 34.5 POSITION VECTOR OF A POINT

We fix an arbitrary point O in space. Given any point P in space, we join it to O to get the vector $\overrightarrow{OP}$. This is called the position vector of the point P with respect to O, called the ***origin of reference***. Thus, to each given point in space there corresponds a unique position vector with respect to a given origin of reference. Conversely, given an origin of reference O, to each vector with the initial point O, corresponds a point namely, its terminal point in space.

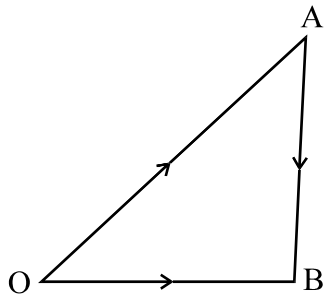


Fig. 34.19

Consider a vector AB. Let O be the origin of reference.

Then $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ or $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

or $\overrightarrow{AB}$ = (Position vector of terminal point B)–(Position vector of initial point A)

## 34.6 MULTIPLICATION OF A VECTOR BY A SCALAR

The product of a non-zero vector $\vec{a}$ by the scalar $x \neq 0$ is a vector whose length is equal to $|x||\vec{a}|$ and whose direction is the same as that of $\vec{a}$ if $x > 0$ and opposite to that of $\vec{a}$ if $x < 0$.

The product of the vector $\vec{a}$ by the scalar x is denoted by $x\vec{a}$.

The product of vector $\vec{a}$ by the scalar 0 is the vector $\vec{\mathbf{0}}$.

By the definition it follows that the product of a zero vector by any non-zero scalar is the zero vector i.e., $x\vec{0} = \vec{0}$ ; also $0\vec{a} = \vec{0}$.

**Laws of multiplication of vectors :** If $\vec{a}$ and $\vec{b}$ are vectors and x, y are scalars, then

(i) $x(y\vec{a}) = (x\,y)\vec{a}$

(ii) $x\vec{a} + y\vec{a} = (x + y)\vec{a}$

(iii) $x\vec{a} + x\vec{b} = x(\vec{a} + \vec{b})$

(iv) $0\vec{a} + x\vec{0} = \vec{0}$

Recall that two collinear vectors have the same direction but may have different magnitudes. This implies that $\vec{a}$ is collinear with a non-zero vector $\vec{b}$ if and only if there exists a number (scalar) x such that

$$\vec{a} = x\vec{b}$$

**Theorem** A necessary and sufficient condition for two vectors $\vec{a}$ and $\vec{b}$ to be collinear is that there exist scalars x and y (not both zero simultaneously) such that $x\vec{a} + y\vec{b} = \vec{0}$.

**The Condition is necessary**

**Proof :** Let $\vec{a}$ and $\vec{b}$ be collinear. Then there exists a scalar $l$ such that $\vec{a} = l\vec{b}$

i.e., $$\vec{a} + (-l)\vec{b} = \vec{0}$$

$\therefore$ We are able to find scalars x ( = 1) and $y\ (= -l)$ such that $x\vec{a} + y\vec{b} = \vec{0}$

Note that the scalar 1 is non-zero.

**The Condition is sufficient**

It is now given that $x\vec{a} + y\vec{b} = \vec{0}$ and $x \neq 0$ and $y \neq 0$ simultaneously.

We may assume that $y \neq 0$

$$\therefore \qquad y\vec{b} = -x\vec{a} \quad \Rightarrow \quad \vec{b} = -\frac{x}{y}\vec{a}$$ i.e., $\vec{b}$ and $\vec{a}$ are collinear.

**Corollary :** Two vectors $\vec{a}$ and $\vec{b}$ are non-collinear if and only if every relation of the form $x\vec{a} + y\vec{b} = \vec{0}$ given as x = 0 and y = 0.

[**Hint :** If any one of x and y is non-zero say y, then we get $\vec{b} = -\frac{x}{y}\vec{a}$ which is a contradiction]

**Example 34.5** Find the number x by which the non-zero vector $\vec{a}$ be multiplied to get

(i) $\hat{a}$ (ii) $-\hat{a}$

**Solution :** (i) $x\vec{a} = \hat{a}$ i.e., $x|\vec{a}|\hat{a} = \hat{a}$

$$\Rightarrow \qquad x = \frac{1}{|\vec{a}|}$$

(ii) $x\vec{a} = -\hat{a}$ i.e., $x|\vec{a}|\hat{a} = -\hat{a}$

$$\Rightarrow \qquad x = -\frac{1}{|\vec{a}|}$$

**Example 34.6** The vectors $\vec{a}$ and $\vec{b}$ are not collinear. Find x such that the vector

$$\vec{c} = (x-2)\vec{a} + \vec{b} \text{ and } \vec{d} = (2x+1)\vec{a} - \vec{b}$$

**Solution :** $\vec{c}$ is non-zero since the co-efficient of $\vec{b}$ is non-zero.

$\therefore$ There exists a number y such that $\vec{d} = y\vec{c}$

i.e. $$(2x+1)\vec{a} - \vec{b} = y(x-2)\vec{a} + y\vec{b}$$

$$\therefore \qquad (yx - 2y - 2x - 1)\vec{a} + (y+1)\vec{b} = 0$$

As $\vec{a}$ and $\vec{b}$ are non-collinear.

$\therefore$ $$yx - 2y - 2x - 1 = 0 \text{ and } y + 1 = 0$$

Solving these we get $y = -1$ and $x = \frac{1}{3}$

Thus $$\vec{c} = -\frac{5}{3}\vec{a} + \vec{b} \text{ and } \vec{d} = \frac{5}{3}\vec{a} - \vec{b}$$

We can see that $\vec{c}$ and $\vec{d}$ are opposite vectors and hence are collinear.

**Example 34.7** The position vectors of two points A and B are $2\vec{a} + 3\vec{b}$ and $3\vec{a} + \vec{b}$ respectively. Find $\overrightarrow{AB}$.

**Solution :** Let O be the origin of reference.

Then $$\overrightarrow{AB} = \text{Position vector of B — Position vector of A}$$
$$= \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (3\vec{a} + \vec{b}) - (2\vec{a} + 3\vec{b})$$
$$= (3 - 2)\vec{a} + (1 - 3)\vec{b} = \vec{a} - 2\vec{b}$$

**Example 34.8** Show that the points P, Q and R with position vectors $\vec{a} - 2\vec{b}$, $2\vec{a} + 3\vec{b}$ and $-7\vec{b}$ respectively are collinear.

**Solution :** $\overrightarrow{PQ}$ = Position vector of Q — Position vector of P
$$= (2\vec{a} + 3\vec{b}) - (\vec{a} - 2\vec{b})$$
$$= \vec{a} + 5\vec{b} \quad \text{....(i)}$$

and $\overrightarrow{QR}$ = Position vector of R — Position vector of Q
$$= -7\vec{b} - (2\vec{a} + 3\vec{b})$$
$$= -7\vec{b} - 2\vec{a} - 3\vec{b}$$
$$= -2\vec{a} - 10\vec{b}$$
$$= -2(\vec{a} + 5\vec{b}) \quad \text{....(ii)}$$

From (i) and (ii) we get $\overrightarrow{PQ} = -2\,\overrightarrow{QR}$, a scalar multiple of $\overrightarrow{QR}$

$\therefore$ $$\overrightarrow{PQ} \,||\, \overrightarrow{QR}$$

But Q is a common point

$\therefore$ $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are collinear. Hence points P, Q and R are collinear.

## CHECK YOUR PROGRESS 34.3

1. The position vectors of the points A and B are $\vec{a}$ and $\vec{b}$ respectively with respect to a given origin of reference. Find $\overrightarrow{AB}$.
2. Interpret each of the following :

   (i) $3\vec{a}$ (ii) $-5\vec{b}$
3. The position vectors of points A, B, C and D are respectively $2\vec{a}$, $3\vec{b}$, $4\vec{a}+3\vec{b}$ and $\vec{a}+2\vec{b}$. Find $\overrightarrow{DB}$ and $\overrightarrow{AC}$.
4. Find the magnitude of the product of a vector $\vec{n}$ by a scalar y.
5. State whether the product of a vector by a scalar is a scalar or a vector.
6. State the condition of collinearity of two vectors $\vec{p}$ and $\vec{q}$.
7. Show that the points with position vectors $5\vec{a}+6\vec{b}$, $7\vec{a}-8\vec{b}$ and $3\vec{a}+20\vec{b}$ are collinear.

## 34.7 CO-PLANARITY OF VECTORS

Given any two non-collinear vectors $\vec{a}$ and $\vec{b}$, they can be made to lie in one plane. There (in the plane), the vectors will be intersecting. We take their common point as O and let the two vectors be $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Given a third vector $\vec{c}$, coplanar with $\vec{a}$ and $\vec{b}$, we can choose its initial point also as O. Let C be its terminal point. With $\overrightarrow{OC}$ as diagonal complete the parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides.

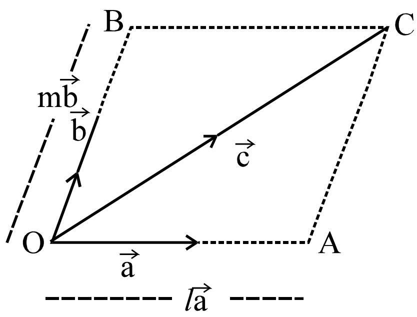



**Fig. 34.20**

$$\therefore \qquad \vec{c} = l\,\vec{a} + m\,\vec{b}$$

Thus, any $\vec{c}$, coplanar with $\vec{a}$ and $\vec{b}$, is expressible as a linear combination of $\vec{a}$ and $\vec{b}$.

i.e. $$\vec{c} = l\,\vec{a} + m\,\vec{b}.$$

## 34.8 RESOLUTION OF A VECTOR ALONG TWO PER PERPEN DICULAR AXES

Consider two mutually perpendicular unit vectors $\hat{i}$ and $\hat{j}$ along two mutually perpendicular axes OX and OY. We have seen above that any vector $\vec{r}$ in the plane of $\hat{i}$ and $\hat{j}$, can be written in the form $\vec{r} = x\hat{i} + y\hat{j}$

If O is the initial point of $\vec{r}$, then OM = x and ON = y and $\overrightarrow{OM}$ and $\overrightarrow{ON}$ are called the component vectors of $\vec{r}$ along x-axis and y-axis.

$\overrightarrow{OM}$ and $\overrightarrow{ON}$, in this special case, are also called the ***resolved parts*** of $\vec{r}$

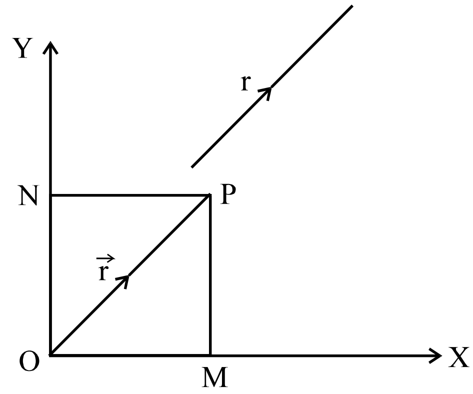



**Fig. 34.21**

## 34.9 RESOLUTION OF A VECTOR IN THREE DIMENSIONS ALONG THREE MUTUALLY PERPENDICULAR AXES

The concept of resolution of a vector in three dimensions along three mutually perpendicular axes is an extension of the resolution of a vector in a plane along two mutually perpendicular axes.

Any vector $\vec{r}$ in space can be expressed as a linear combination of three mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ as is shown in the adjoining Fig. 14.22.

We complete the rectangular parallelopiped with $\overrightarrow{OP} = \vec{r}$ as its diagonal :

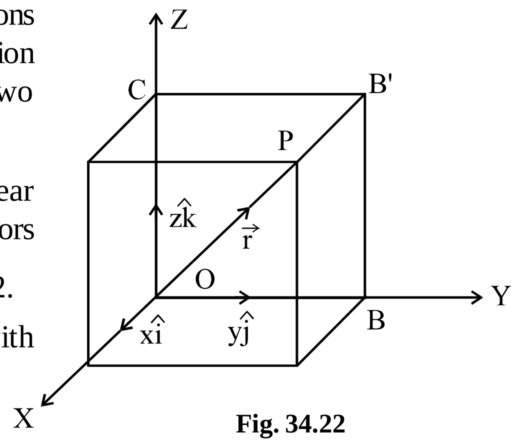



**Fig. 34.22**

then $\qquad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the resolved parts of $\vec{r}$ along three mutually perpendicular axes.

Thus any vector $\vec{r}$ in space is expressible as a linear combination of three mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Refer to Fig. 34.21 in which $OP^2 = OM^2 + ON^2$ (Two dimensions)

or $\qquad \vec{r}^{\,2} = x^2 + y^2 \qquad$ ......(i)

and in Fig. 34.22

$$OP^2 = OA^2 + OB^2 + OC^2$$

$$\vec{r}^{\,2} = x^2 + y^2 + z^2 \qquad \text{.......(ii)}$$

Magnitude of $\vec{r} = |\vec{r}|$ in case of $\qquad$ (i) is $\sqrt{x^2 + y^2}$

and $\qquad$ (ii) is $\sqrt{x^2 + y^2 + z^2}$

***Note :*** Given any three non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ (not necessarily mutually perpendicular unit vectors) any vector $\vec{d}$ is expressible as a linear combination of

$$\vec{a},\ \vec{b} \text{ and } \vec{c}, \text{ i.e., } \vec{d} = x\,\vec{a} + y\,\vec{b} + z\,\vec{c}$$

**Example 34.9** A vector of 10 Newton is 30° north of east. Find its components along east and north directions.

**Solution :** Let $\hat{i}$ and $\hat{j}$ be the unit vectors along $\overrightarrow{OX}$ and $\overrightarrow{OY}$ (East and North respectively)

Resolve OP in the direction OX and OY.

$$\therefore \quad \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{ON}$$

$$= 10 \cos 30°\ \hat{i} + 10 \sin 30°\ \hat{j}$$

$$= 10.\frac{\sqrt{3}}{2}\hat{i} + 10.\frac{1}{2}\hat{j}$$

$$= 5\sqrt{3}\ \hat{i} + 5\ \hat{j}$$

$\therefore$ Component along (i) East $= 5\sqrt{3}$ Newton

(ii) North = 5 Newton

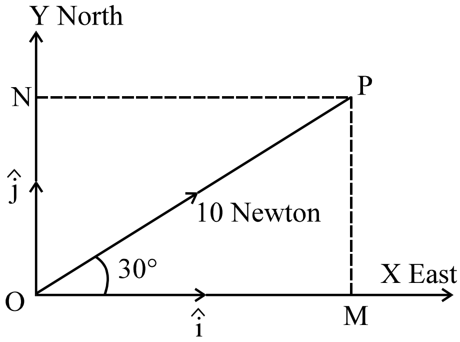



**Fig. 34.23**

**Example 34.10** Show that the following vectors are coplanar :

$$\vec{a} - 2\vec{b},\ 3\vec{a} + \vec{b} \text{ and } \vec{a} + 4\vec{b}$$

**Solution :** The vectors will be coplanar if there exists scalars x and y such that

$$\vec{a} + 4\vec{b} = x(\vec{a} - 2\vec{b}) + y(3\vec{a} + \vec{b})$$

$$= (x + 3y)\,\vec{a} + (-2x + y)\,\vec{b} \qquad \text{.....(i)}$$

Comparing the co-efficients of $\vec{a}$ and $\vec{b}$ on both sides of (i), we get

$$x + 3y = 1 \text{ and } -2x + y = 4$$

which on solving, gives $x = -\dfrac{11}{7}$ and $y = \dfrac{6}{7}$

As $\vec{a} + 4\vec{b}$ is expressible in terms of $\vec{a} - 2\vec{b}$ and $3\vec{a} + \vec{b}$, hence the three vectors are coplanar.

**Example 34.11** Given $\vec{r_1} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{r_2} = 2\hat{i} - 4\hat{j} - 3\hat{k}$, find the magnitudes of

(a) $\vec{r_1}$ (b) $\vec{r_2}$ (c) $\vec{r_1} + \vec{r_2}$ (d) $\vec{r_1} - \vec{r_2}$

**Solution :**

(a) $$|\vec{r_1}| = |\hat{i} - \hat{j} + \hat{k}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

(b) $\quad |\vec{r}_2| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}$

(c) $\quad \vec{r}_1 + \vec{r}_2 = (\hat{i} - \hat{j} + \hat{k}) + (2\hat{i} - 4\hat{j} - 3\hat{k}) = 3\hat{i} - 5\hat{j} - 2\hat{k}$

$\therefore \quad |\vec{r}_1 + \vec{r}_2| = |3\hat{i} - 5\hat{j} - 2\hat{k}| = \sqrt{3^2 + (-5)^2 + (-2)^2} = \sqrt{38}$

(d) $\quad \vec{r}_1 - \vec{r}_2 = (\hat{i} - \hat{j} + \hat{k}) - (2\hat{i} - 4\hat{j} - 3\hat{k}) = -\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore \quad |\vec{r}_1 - \vec{r}_2| = |-\hat{i} + 3\hat{j} + 4\hat{k}| = \sqrt{(-1)^2 + 3^2 + 4^2} = \sqrt{26}$

**Example 34.12** Determine the unit vector parallel to the resultant of two vectors

$$\vec{a} = 3\hat{i} + 2\hat{j} - 4\hat{k} \text{ and } \vec{b} = \hat{i} + \hat{j} + 2\hat{k}$$

**Solution :** The resultant vector $\vec{R} = \vec{a} + \vec{b} = (3\hat{i} + 2\hat{j} - 4\hat{k}) + (\hat{i} + \hat{j} + 2\hat{k})$

$$= 4\hat{i} + 3\hat{j} - 2\hat{k}$$

Magnitude of the resultant vector $\vec{R}$ is $|\vec{R}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$

$\therefore$ The unit vector parallel to the resultant vector

$$\frac{\vec{R}}{|\vec{R}|} = \frac{1}{\sqrt{29}}(4\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{4}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}$$

**Example 34.13** Find a unit vector in the direction of $\vec{r} - \vec{s}$

where $\vec{r} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{s} = 2\hat{i} - \hat{j} + 2\hat{k}$

**Solution :** $\vec{r} - \vec{s} = (\hat{i} + 2\hat{j} - 3\hat{k}) - (2\hat{i} - \hat{j} + 2\hat{k})$

$$= -\hat{i} + 3\hat{j} - 5\hat{k}$$

$\therefore \quad |\vec{r} - \vec{s}| = \sqrt{(-1)^2 + (3)^2 + (-5)^2} = \sqrt{35}$

$\therefore$ Unit vector in the direction of $(\vec{r} - \vec{s})$

$$= \frac{1}{\sqrt{35}}(-\hat{i} + 3\hat{j} - 5\hat{k}) = -\frac{1}{\sqrt{35}}\hat{i} + \frac{3}{\sqrt{35}}\hat{j} - \frac{5}{\sqrt{35}}\hat{k}$$

**Example 34.14** Find a unit vector in the direction of $2\vec{a} + 3\vec{b}$ where $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} - \hat{k}$.

**Solution :** $2\vec{a} + 3\vec{b} = 2(\hat{i} + 3\hat{j} + \hat{k}) + 3(3\hat{i} - 2j - \hat{k})$

$$= (2\hat{i} + 6\hat{j} + 2\hat{k}) + (9\hat{i} - 6j - 3\hat{k})$$

$$= 11\hat{i} - \hat{k}.$$

$\therefore \quad |2\vec{a}+3\vec{b}| = \sqrt{(11)^2+(-1)^2} = \sqrt{122}$

$\therefore$ Unit vector in the direction of $(2\vec{a}+3\vec{b})$ is $\dfrac{11}{\sqrt{122}}\hat{i} - \dfrac{1}{\sqrt{122}}\hat{k}$ .

**Example 34.15** Show that the following vectors are coplanar :

$4\vec{a}-2\vec{b}-2\vec{c}$, $-2\vec{a}+4\vec{b}-2\vec{c}$ and $-2\vec{a}-2\vec{b}+4\vec{c}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors.

**Solution :** If these vectors be co-planar, it will be possible to express one of them as a linear combination of other two.

Let $\quad -2\vec{a}-2\vec{b}+4\vec{c} = x\,(4\vec{a}-2\vec{b}-2\vec{c}) + y\,(-2\vec{a}+4\vec{b}-2\vec{c})$

where x and y are scalars,

Comparing the co-efficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ from both sides, we get

$$4x-2y=-2,\ -2x+4y=-2 \text{ and } -2x-2y=4$$

These three equations are satisfied by $x=-1,\ y=-1$ Thus,

$$-2\vec{a}-2\vec{b}+4\vec{c} = (-1)\,(4\vec{a}-2\vec{b}-2\vec{c}) + (-1)\,(-2\vec{a}+4\vec{b}-2\vec{c})$$

Hence the three given vectors are co-planar.

## CHECK YOUR PROGRESS 34.4

1. Write the condition that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are co-planar.

2. Determine the resultant vector $\vec{r}$ whose components along two rectangular Cartesian co-ordinate axes are 3 and 4 units respectively.

3. In the adjoining figure :

   $|OA| = 4$, $|OB| = 3$ and

   $|OC| = 5$. Express OP in terms of its component vectors.

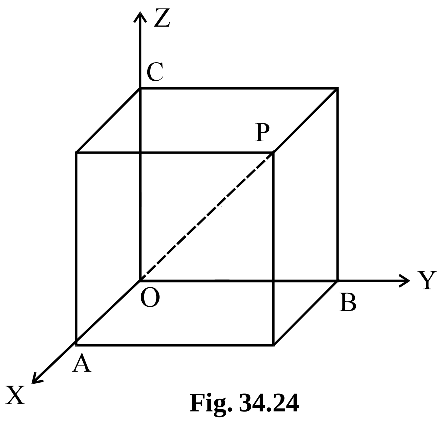


**Fig. 34.24**

4. If $\vec{r_1} = 4\hat{i}+\hat{j}-4\hat{k}$, $\vec{r_2} = -2\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{r_3} = \hat{i}+3\hat{j}-\hat{k}$ then show that

   $|\vec{r_1}+\vec{r_2}+\vec{r_3}| = 7$ ·

5. Determine the unit vector parallel to the resultant of vectors :

   $\vec{a} = 2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$

6. Find a unit vector in the direction of vector $3\vec{a} - 2\vec{b}$ where $\vec{a} = \hat{i} - \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$.

7. Show that the following vectors are co-planar :

   $3\vec{a} - 7\vec{b} - 4\vec{c}$, $3\vec{a} - 2\vec{b} + \vec{c}$ and $\vec{a} + \vec{b} + 2\vec{c}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors.

## 34.10 SECTION FORMULA

Recall that the position vector of a point P is space with respect to an origin of reference O is $\vec{r} = \overrightarrow{OP}$.
In the following, we try to find the position vector of a point dividing a line segment joining two points in the ratio m : n internally.

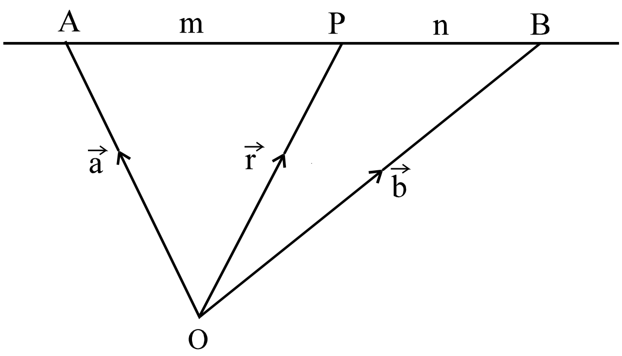



**Fig. 34.25**

Let A and B be two points and $\vec{a}$ and $\vec{b}$ be their position vectors w.r.t. the origin of reference O, so that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.
Let P divide AB in the ratio m : n so that

$$\frac{AP}{PB} = \frac{m}{n} \quad \text{or,} \quad n\overrightarrow{AP} = m\overrightarrow{PB} \qquad \text{....(i)}$$

Since $\quad n\overrightarrow{AP} = m\overrightarrow{PB}$, it follows that

$$n(\overrightarrow{OP} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OP})$$

or $\quad (m+n)\overrightarrow{OP} = m\overrightarrow{OB} + n\overrightarrow{OA}$

or $\quad \overrightarrow{OP} = \dfrac{m\overrightarrow{OB} + n\overrightarrow{OA}}{m+n}$

or $\quad \vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

where $\vec{r}$ is the position vector of P with respect to O.

**Corollary 1 :** If $\frac{m}{n} = 1 \Rightarrow m = n$, then P becomes mid-point of AB.

$\therefore$ The position vector of the mid-point of the join of two given points, whose position vectors are $\vec{a}$ and $\vec{b}$, is given by $\frac{1}{2}(\vec{a} + \vec{b})$.

**Corollary 2 :** The position vector P can also be written as

$$\vec{r} = \frac{\vec{a} + \frac{m}{n}\vec{b}}{1 + \frac{m}{n}} = \frac{\vec{a} + k\,\vec{b}}{1 + k}, \qquad \text{.....(ii)}$$

where $\quad k = \frac{m}{n},\ k \neq -1.$

(ii) represents the position vector of a point which divides the join of two points with position vectors $\vec{a}$ and $\vec{b}$, in the ratio k : 1.

**Corollary 3 :** The position vector of a point P which divides AB in the ratio m : n externally is

$$\vec{r} = \frac{n\,\vec{a} - m\,\vec{b}}{n - m}$$ [**Hint :** This division is in the ratio –m : n]

**Example 34.16** Find the position vector of a point which divides the join of two points whose position vectors are given by $\vec{x}$ and $\vec{y}$ in the ratio 2 : 3 internally.

**Solution :** Let $\vec{r}$ be the position vector of the point.

$$\therefore \qquad \vec{r} = \frac{3\,\vec{x} + 2\,\vec{y}}{3 + 2} = \frac{1}{5}(3\,\vec{x} + 2\,\vec{y}).$$

**Example 34.17** Find the position vector of mid-point of the line segment AB, if the position vectors of A and B are respectively, $\vec{x} + 2\,\vec{y}$ and $2\,\vec{x} - \vec{y}$.

**Solution :** Position vector of mid-point of AB

$$= \frac{(\vec{x} + 2\,\vec{y}) + (2\,\vec{x} - \vec{y})}{2}$$

$$= \frac{3}{2}\vec{x} + \frac{1}{2}\vec{y}$$

**Example 34.18** The position vectors of vertices A, B and C of $\Delta ABC$ are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Find the position vector of the centroid of $\Delta ABC$.

**Solution :** Let D be the mid-point of side BC of $\Delta ABC$.

Let G be the centroid of $\Delta ABC$. Then G divides AD in the ratio 2 : 1 i.e. AG : GD = 2 : 1.

Now position vector of D is $\dfrac{\vec{b}+\vec{c}}{2}$

$\therefore$ Position vector of G is $\dfrac{2\cdot\dfrac{\vec{b}+\vec{c}}{2}+1\cdot\vec{a}}{2+1}$

$$= \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

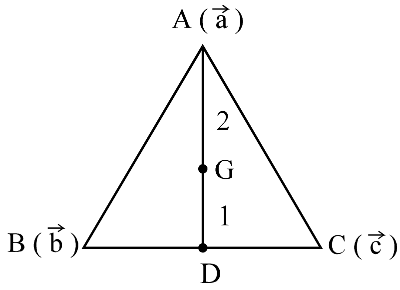



**Fig. 34.26**

## CHECK YOUR PROGRESS 34.5

1. Find the position vector of the point C if it divides AB in the ratio (i) $\frac{1}{2} : \frac{1}{3}$

   (ii) 2 : – 3, given that the position vectors of A and B are $\vec{a}$ and $\vec{b}$ respectively.
2. Find the point which divides the join of $P(\vec{p})$ and $Q(\vec{q})$ internally in the ratio 3 : 4.
3. CD is trisected at points P and Q. Find the position vectors of points of trisection, if the position vectors of C and D are $\vec{c}$ and $\vec{d}$ respectively
4. Using vectors, prove that the medians of a triangle are concurrent.
5. Using vectors, prove that the line segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.

## 34.11 DIRECTION COSINES OF A VECTOR

In the adjoining figure $\overrightarrow{AB}$ is a vector in the space and $\overrightarrow{OP}$ is the position vector of the point P($x$, $y$, $z$) such that $\overrightarrow{OP} \parallel \overrightarrow{AB}$. Let $\overrightarrow{OP}$ makes angles $\alpha$, $\beta$ and $\gamma$ respectively with the positive directions of $x$, $y$ and $z$ axis respectively. $\alpha$, $\beta$ and $\gamma$ are called direction angles of vector $\overrightarrow{OP}$ and $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called its direction cosines.

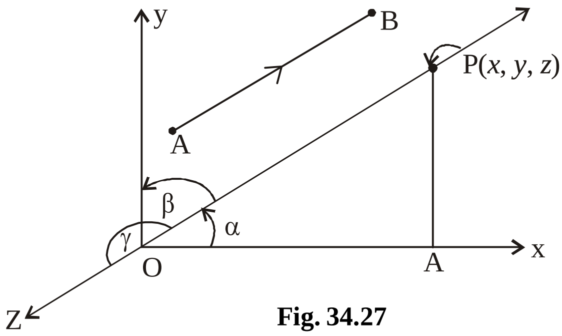



**Fig. 34.27**

Since $\overrightarrow{OP} \parallel \overrightarrow{AB}$, therefore $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are direction cosines of vector $\overrightarrow{AB}$ also.

Direction cosines of a vector are the cosines of the angles subtended by the vector with the positive directions of $x$, $y$ and $z$ axes respectively.

- By reversing the direction, we observe that $\overrightarrow{PO}$ makes angles $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$ with the positive directions of $x$, $y$ and $z$ axes respectively. So $\cos(\pi - \alpha) = -\cos\alpha$, $\cos(\pi - \beta) = -\cos\beta$ and $\cos(\pi - \gamma) = -\cos\gamma$ are the direction cosines of $\overrightarrow{PO}$. In fact any vector in space can be extended in two directions so it has two sets of direction cosines. If $(\cos\alpha, \cos\beta, \cos\gamma)$ is one set of direction cosines then $(-\cos\alpha, -\cos\beta, -\cos\gamma)$ is the other set. It is enough to mention any one set of direction cosines of a vector.
- Direction cosines of a vector are usually denoted by $l$, $m$ and $n$. In other words $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.
- Since $\overrightarrow{OX}$ makes angles 0°, 90° and 90° with $\overrightarrow{OX}$, $\overrightarrow{OY}$ and $\overrightarrow{OZ}$ respectively. Therefore cos0°, cos90°, cos90° i.e. 1, 0, 0 are the direction cosines of x-axis. Similarly direction cosines of $y$ and $z$ axes are (0, 1, 0) and (0, 0, 1) respectively.

In the figure, 1 let $|\overrightarrow{OP}| = r$. and $PA \perp OX$.

Now in right angled $\Delta OAP$, $\frac{OA}{OP} = \cos\alpha$

i.e. $\quad OA = OP\cos\alpha$

i.e. $\quad x = r.l \Rightarrow \quad x = l\,r$

Similarly by dropping perpendiculars to $y$ and $z$ axes respectively we get $y = mr$ and $z = nr$.

Now $\quad x^2 + y^2 + z^2 = r^2(l^2 + m^2 + n^2) \quad$ ...(i)

But $\quad |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

or $\quad |\overrightarrow{OP}|^2 = x^2 + y^2 + z^2$

$\quad = r^2$

therefore from (i) $l^2 + m^2 + n^2 = 1$

Again $l = \frac{x}{r}$, $m = \frac{y}{r}$, $n = \frac{z}{r}$

i.e. $\quad l = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$, $m = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$, $n = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

Hence, if $P(x, y, z)$ is a point in the space, then direction cosines of $\overrightarrow{OP}$ are $\frac{x}{\sqrt{x^2 + y^2 + z^2}}$, $\frac{y}{\sqrt{x^2 + y^2 + z^2}}$, $\frac{z}{\sqrt{x^2 + y^2 + z^2}}$

## 34.11.1 DIRECTION COSINES OF A VECTOR JOINING TWO POINTS :

In the adjoining figure $\overrightarrow{PQ}$ is a vector joining points P($x_1$ $y_1$ z) and Q($x_2$ $y_2$ $z_2$). If we shift the origin to the point P($x_1y_1z_1$) without changing the direction of coordinate axes. The coordinates of point Q becomes ($x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$) therefore direction cosines of

$\overrightarrow{PQ}$ are $\dfrac{x_2 - x_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$, $\dfrac{y_2 - y_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$,

$\dfrac{z_2 - z_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$,

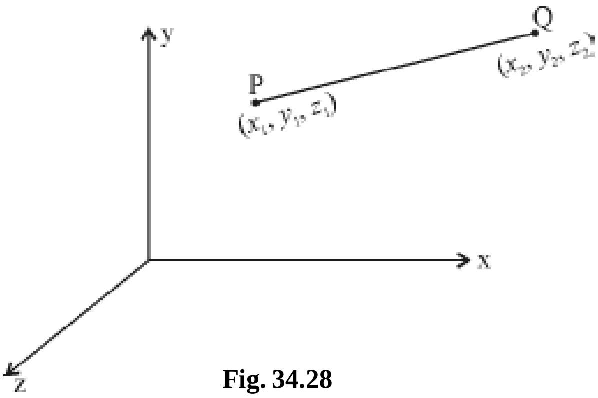



**Fig. 34.28**

## 34.11.2 DIRECTION RATIOS OF A VECTOR :

Any three real numbers which are proportional to the direction cosines of a vector are called direction ratios of that vector. Let *l*, *m*, *n* be the direction cosines of a vector and *a*, *b*, *c* be the direction ratios.

then, $\dfrac{a}{l} = \dfrac{b}{m} = \dfrac{c}{n} = \lambda$ (say)

$\Rightarrow \quad a = \lambda l, b = \lambda m, c = \lambda n$

$\therefore \quad a^2 + b^2 + c^2 = \lambda^2(l^2 + m^2 + n^2)$

$\Rightarrow \quad \lambda^2 = a^2 + b^2 + c^2$

i.e. $\lambda = \pm\sqrt{a^2 + b^2 + c^2}$

$\therefore \quad l = \pm\dfrac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\dfrac{b}{\sqrt{a^2+b^2+c^2}}, n = \dfrac{\pm c}{\sqrt{a^2+b^2+c^2}}$

- If $a$, $b$, $c$ are direction ratios of a vector then for every $\lambda \neq 0$, $\lambda a$, $\lambda b$, $\lambda c$ are also its direction ratios. Thus a vector can have infinite number of direction ratios.
- If P($x,y,z$) is a point in the space, then the direction ratios of $\overrightarrow{OP}$ are $x$, $y$, $z$.
- If P($x_1$, $y_1$, $z_1$) and Q($x_2$, $y_2$, $z_2$) are two points in the space then the direction ratios of $\overrightarrow{PQ}$ are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.
- $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$ in general.

**Example 34.19** Let P be a point in space such that OP = $\sqrt{3}$ and $\overrightarrow{OP}$ makes angles $\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{3}$ with positve directions of, $x$, $y$ and $z$ axes respectively. Find coordinates of point P.

**Solution :** d.c.s of $\overrightarrow{OP}$ are $\cos\frac{\pi}{3}, \cos\frac{\pi}{4}, \cos\frac{\pi}{3}$ i.e. $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$

$\therefore$ coordinates of point are $x = lr = \frac{1}{2} \times \sqrt{3} = \frac{\sqrt{3}}{2}$

$$y = mr = \frac{1}{\sqrt{2}} \times \sqrt{3} = \frac{\sqrt{3}}{\sqrt{2}}$$

and $$z = nr = \frac{1}{2} \times \sqrt{3} = \frac{\sqrt{3}}{2}$$

**Example 34.20** If P(1, 2, –3) is a point in the space, find the direction cosines of vector $\overrightarrow{OP}$.

**Solution :**

$$l = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{\sqrt{14}}$$

$$m = \frac{y}{\sqrt{x^2 + y^2 + z^2}} = \frac{2}{\sqrt{14}}$$

$$n = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{-3}{\sqrt{14}}$$

**Example 34.21** Can $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ be direction cosines of a vector.

**Solution :** $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{4}{3} \neq 1$

$\therefore$ $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ can not be direction cosines of a vector.

**Example 34.22** If P(2, 3, –6) and Q(3, –4, 5) are two points in the space. Find the direction cosines of $\overrightarrow{OP}, \overrightarrow{QO}$ and $\overrightarrow{PQ}$, where O is the origin.

**Solution :** D.C.'S of $\overrightarrow{OP}$ are $\dfrac{2}{\sqrt{2^2+3^2+(-6)^2}}, \dfrac{3}{\sqrt{2^2+3^2+(-6)^2}}, \dfrac{-6}{\sqrt{2^2+3^2+(-6)^2}}$

i.e. $\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{-6}{7}$.

Similarly d.c.'s of $\overrightarrow{QO}$ are $\dfrac{-3}{5\sqrt{2}}, \dfrac{4}{5\sqrt{2}}, \dfrac{-5}{\sqrt{2}}$

D.C.'s of $\overrightarrow{PQ}$ are : $\dfrac{3-2}{\sqrt{(3-2)^2+(-4-3)^2+(5+6)^2}}, \dfrac{-4-3}{\sqrt{(3-2)^2+(-4-3)^2+(5+6)^2}}$

$\dfrac{5+6}{\sqrt{(3-2)^2+(-4-3)^2+(5+6)^2}}$

i.e. $\dfrac{1}{\sqrt{171}}, \dfrac{-7}{\sqrt{171}}, \dfrac{11}{\sqrt{171}}$

**Example 34.23** Find the direction cosines of a vector which makes equal angles with the axes.

**Solution :** Suppose the given vector makes angle $\alpha$ with each of the $\overrightarrow{OX}, \overrightarrow{OY}$ and $\overrightarrow{OZ}$. Therefore $\cos\alpha$, $\cos\alpha$, $\cos\alpha$ are the direction cosines of the vector.

Now, $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

i.e. $\cos\alpha = \pm\dfrac{1}{\sqrt{3}}$

$\therefore$ d.c.'s of the vector are $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

**Example 34.24** If P(1, 2, –3) and Q(4, 3, 5) are two points in space, find the direction ratios of $\overrightarrow{OP}, \overrightarrow{QO}$ and $\overrightarrow{PQ}$

**Solution :** d.r.'s of $\overrightarrow{OP}$ are 1, 2, –3

d.r.'s of $\overrightarrow{QO}$ are (–4, –3, –5) or (4, 3, 5)

d.r.'s of $\overrightarrow{PQ}$ are 4 –1, 3 –2, 5 – (–3)

i.e. 3, 1, 8.

## CHECK YOUR PROGRESS 34.6

1. Fill in the blanks:
   (i) Direction cosines of y-axis are...
   (ii) If $l$, $m$, $n$ are direction cosines of a vector, then $l^2 + m^2 + n^2 = ....$

(iii) If a, b, c are direction ratios of a vector, then $a^2 + b^2 + c^2$ is .... to 1

(iv) The direction cosines of a vector which makes equal angles with the coordinate axes are...

(v) If two vectors are parallel to each other then their direction ratios are...

(vi) (1, –1, 1) are not direction cosines of any vector because...

(vii) The number of direction ratios of a vector are... (finite/infinite)

2. If P(3, 4, –5) is a point in the space. Find the direction cosines of $\overrightarrow{OP}$.
3. Find the direction cosines of $\overrightarrow{AB}$ where A(–2, 4, –5) and B(1, 2, 3) are two points in the space.
4. If a vector makes angles 90°, 135° and 45° with the positive directions of $x$, $y$ and $z$ axis respectively, find its direction ratios.

## 34.12 PRODUCT OF VECTORS

In Section 34.9, you have multiplied a vector by a scalar. The product of vector with a scalar gives us a vector quantity. In this section we shall take the case when a vector is multiplied by another vector. There are two cases :

(i) When the product of two vectors is a scalar, we call it a scalar product, also known as dot product corresponding to the symbol '$\bullet$' used for this product.

(ii) When the product of two vectors is a vector, we call it a vector product, also known as cross product corresponding to the symbol '$\times$' used for this product.

## 34.13 SCALAR PRODUCT OF TWO VECTORS

Let $\vec{a}$ and $\vec{b}$ two vectors and $\theta$ be the angle between them. The scalar product, denoted by $\vec{a}$, $\vec{b}$, is defined by

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

Clearly, $\vec{a}\cdot\vec{b}$ is a scalar as $|\vec{a}|$, $|\vec{b}|$ and $\cos\theta$ are all scalars.

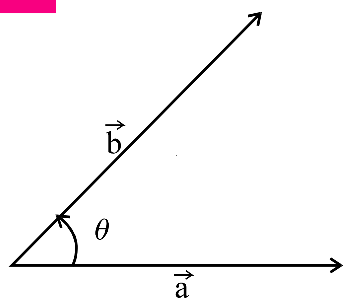


**Fig. 34.29**

### Remarks

1. If $\vec{a}$ and $\vec{b}$ are like vectors, then $\vec{a}\cdot\vec{b} = ab\cos\theta = ab$, where a and b are magnitudes of $\vec{a}$ and $\vec{b}$.
2. If $\vec{a}$ and $\vec{b}$ are unlike vectors, then $\vec{a}\cdot\vec{b} = ab\cos\pi = -ab$
4. Angle $\theta$ between the vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$
5. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ and $\vec{a}\cdot(\vec{b}+\vec{c}) = (\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c})$.
6. $n(\vec{a}\cdot\vec{b}) = (n\vec{a})\cdot\vec{b} = \vec{a}\cdot(n\vec{b})$ where n is any real number.
7. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ as $\hat{i}$, $\hat{j}$ and $\hat{k}$ are mutually perpendicular unit vectors.

**Example 34.25** If $\vec{a} = 3\hat{i} + 2\hat{j} - 6\hat{k}$ and $\vec{b} = 4\hat{i} - 3\hat{j} + \hat{k}$, find $\vec{a} \cdot \vec{b}$.

Also find angle between $\vec{a}$ and $\vec{b}$.

**Solution :** $\vec{a} \cdot \vec{b} = (3\hat{i} + 2\hat{j} - 6\hat{k}) \cdot (4\hat{i} - 3\hat{j} + \hat{k})$

$$= 3 \times 4 + 2 \times (-3) + (-6) \times 1$$

$$\left[\because\ \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \text{ and } \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0\right]$$

$$= 12 - 6 - 6 = 0$$

Let $\theta$ be the angle between the vectors $\vec{a}$ and $\vec{b}$

Then $$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = 0$$

$\therefore$ $$\theta = \frac{\pi}{2}.$$

## 34.14 VECTOR PRODUCT OF TWO VECTORS

Before we define vector product of two vectors, we discuss below right handed and left handed screw and associate it with corresponding vector triad.

### 34.14.1 Right Handed Screw

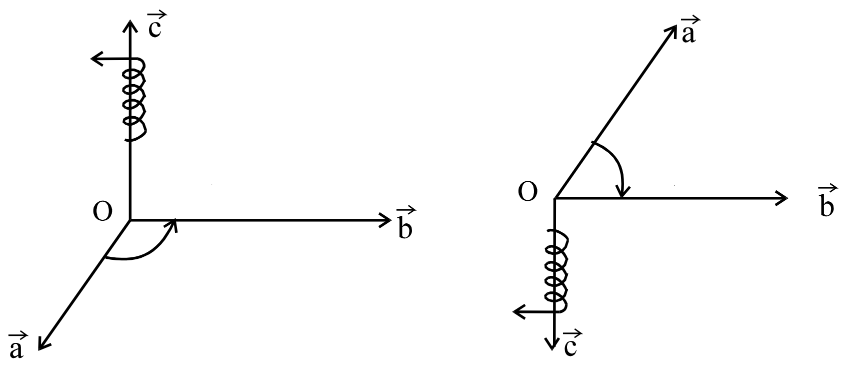
If a screw is taken and rotated in the anticlockwise direction, it translates towards the reader. It is called ***right handed screw***.

### 34.14.2 Left handed Screw

If a screw is taken and rotated in the clockwise direction, it translates away from the reader. It is called a left handed screw.

Now we associate a screw with given ordered vector triad.

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors whose initial point is O.




(i) (ii)

**Fig. 34.30**

Now if a right handed screw at O is rotated from $\vec{a}$ towards $\vec{b}$ through an angle <180°, it will undergo a translation along $\vec{c}$ [Fig. 34.28 (i)]

Similarly if a left handed screw at O is rotated from $\vec{a}$ to $\vec{b}$ through an angle <180°, it will undergo a translation along $\vec{c}$ [Fig. 34.28 (ii)]. This time the direction of translation will be opposite to the first one.

Thus an ordered vector triad $\vec{a}$, $\vec{b}$, $\vec{c}$ is said to be right handed or left handed according as the right handed screw translated along $\vec{c}$ or opposite to $\vec{c}$ when it is rotated through an angle less than 180°.

## 34.14.3 VECTOR (CROSS) PRODUCT OF THE VECTORS :

If $\vec{a}$ and $\vec{b}$ are two non zero vectors then their cross product is denoted by $\vec{a} \times \vec{b}$ and defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta . \hat{n}$

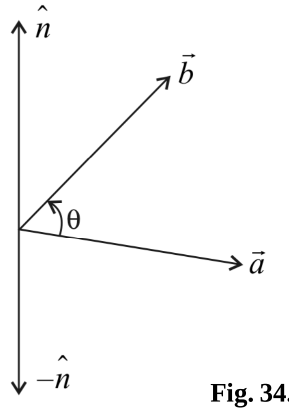



**Fig. 34.31**

Where $\theta$ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$; such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system (see figure) i.e. the right handed system rotated from $\vec{a}$ to $\vec{b}$ moves in the direction of $\hat{n}$.

- $\vec{a} \times \vec{b}$ is a vector and $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.
- If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then $\theta$ is not defined and in that case we consider $\vec{a} \times \vec{b} = \vec{0}$.
- If $\vec{a}$ and $\vec{b}$ are non zero vectors. Then $\vec{a} \times \vec{b} = \vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are collinear or parallel vectors. i.e. $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \| \vec{b}$.
- In particular $\vec{b} \times \vec{b} = \vec{0}$ and $\vec{b} \times (-\vec{b}) = \vec{0}$ because in the first situation $\theta = 0$ and in 2nd case $\theta = \pi$. Making the value of $\sin\theta = 0$ in both the cases.
- $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$
- $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$ and $\hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$.

- $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}.$
- $\lambda(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$.
- Angle $\theta$ between two vectors $\vec{a}$ and $\vec{b}$ is given as

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}.$$

- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle then its area is given by $\frac{1}{2}|\vec{a}\times\vec{b}|$.
- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given by $|\vec{a}\times\vec{b}|$
- If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\hat{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then

$$\hat{a}\times\hat{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$= (a_2b_3-a_3b_2)\hat{i}-(a_1b_3-a_3b_1)\hat{j}+(a_1b_2-a_2b_1)\hat{k}$$

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

**Example 34.26** Using cross product find the angle between the vectors $\vec{a}=2\hat{i}+\hat{j}-3\hat{k}$ and $\vec{b} = 3\hat{i}-2\hat{j}+\hat{k}$.

**Solution :**

$$\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -3\\ 3 & -2 & 1\end{vmatrix} = \hat{i}(1-6)-\hat{j}(2+9)+\hat{k}(-4-3)$$

$$= -5\hat{i}-11\hat{j}-7\hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{25+121+49}=\sqrt{195}$$

$$|\vec{a}| = \sqrt{4+1+9}=\sqrt{14}$$

$$|\vec{b}| = \sqrt{9+4+1}=\sqrt{14}$$

$$\therefore \quad \sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{195}}{\sqrt{14}.\sqrt{14}}=\frac{\sqrt{195}}{14}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{\sqrt{195}}{14}\right)$$

**Example 34.27** Find a unit vector perpendicular to each of the vectors $\vec{a} = 3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$.

**Solution :**

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -3 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(-2+3) - \hat{j}(-3+3) + \hat{k}(3-2)$$

$$\vec{a} \times \vec{b} = \hat{i} + \hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{1+1} = \sqrt{2}$$

Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ $= \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$$= \frac{\hat{i} + \hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$$

**Example 34.28** Find the area of the triangle having point A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1) as its vertices.

**Solution :**

$$\overrightarrow{AB} = (1-1)\hat{i} + (2-1)\hat{j} + (3-1)\hat{k}$$

$$= \hat{j} + 2\hat{k}$$

$$\overrightarrow{AC} = (2-1)\hat{i} + (3-1)\hat{j} + (1-1)\hat{k}$$

$$= \hat{i} + 2\hat{j}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = \hat{i}(0-4) - \hat{j}(0-2) + \hat{k}(0-1)$$

$$= -4\hat{i} + 2\hat{j} - \hat{k}$$

$$\therefore \quad |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-4)^2 + (2)^2 + (-1)^2} = \sqrt{16+4+1} = \sqrt{21}$$

Hence, Area of $\Delta ABC = \dfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \dfrac{\sqrt{21}}{2}$ unit$^2$.

**Example 34.29** Find the area of the parallelogram having A(5, –1, 1), B(–1, –3, 4), C(1, –6, 10) and D(7, –4, 7) as its vertices.

**Solution :**

$$\overrightarrow{AB} = (-1-5)\hat{i} + (-3+1)\hat{j} + (4-1)\hat{k}$$

$$= -6\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{AD} = (7-5)\hat{i} + (-4+1)\hat{j} + (7-1)\hat{k}$$

$$= 2\hat{i} - 3\hat{j} + 6\hat{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & -2 & 3 \\ 2 & -3 & 6 \end{vmatrix} = \hat{i}(-12+9) - \hat{j}(-36-6) + \hat{k}(18+4)$$

$$= -3\hat{i} + 42\hat{j} + 22\hat{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{9+1764+484} = \sqrt{2257} \text{ unit}^2.$$

## CHECK YOUR PROGRESS 34.7

1. (i) If $\vec{a} \times \vec{b}$ is a unit vector and $|\vec{a}| = 3, |\vec{b}| = \frac{\sqrt{2}}{3}$, then the angle between $\vec{a}$ and $\vec{b}$ is ...

   (ii) If $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$, then angle between $\vec{a}$ and $\vec{b}$ is ...

   (iii) The value of $\hat{i}.(\hat{j} \times \hat{k}) + \hat{j}.(\hat{i} \times \hat{k}) + \hat{k}.(\hat{i} \times \hat{j})$ is ...

2. Find a unit vector perpendicular to both the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

3. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

4. If $\vec{a} = 2\hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$, $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + j\vec{b}$ is perpendicular to $\vec{c}$, find the value of j.

## 34.15 SCALAR TRIPLE PRODUCT :

If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors then the scalar product of $\vec{a} \times \vec{b}$ with $\vec{c}$ is called scalar triple product i.e. $(\vec{a} \times \vec{b}).\vec{c}$ is called scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$. It is usually denoted as $[\vec{a}.\vec{b}\vec{c}]$

- $[\vec{a}\,\vec{b}\,\vec{c}]$ is a scalar quantity.
- $(\vec{a}\times\vec{b}).\vec{c}$ represents the volume of a parallelopiped having $\vec{a},\vec{b},\vec{c}$ as coterminous edges.
- $(\vec{a}\times\vec{b}).\vec{c}=0$ if $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar vectors or any two of the three vectors are equal or parallel.
- In the scalar triple product the position of dot and cross can be interchanged provided the cyclic order of the vectors is maintained i.e.

$$(\vec{a}\times\vec{b}).\vec{c} = \vec{a}\,.(\vec{b}\times\vec{c})$$

$$(\vec{b}\times\vec{c}).\vec{a} = \vec{b}\,.(\vec{c}\times\vec{a})$$

$$(\vec{c}\times\vec{a}).\vec{b} = \vec{c}\,.(\vec{a}\times\vec{b})$$

- $(\vec{a}\times\vec{b})\,.\,\vec{c}=(\vec{b}\times\vec{c})\,.\,\vec{a}=(\vec{c}\times\vec{a})\,.\,\vec{b}$
- $(\vec{a}\times\vec{b})\,.\,\vec{c}=-(\vec{b}\times\vec{a})\,.\,\vec{c}=-\vec{c}\,.\,(\vec{b}\times\vec{a})$

  $(\vec{b}\times\vec{c})\,.\,\vec{a}=-(\vec{c}\times\vec{b})\,.\,\vec{a}=-\vec{a}\,.\,(\vec{c}\times\vec{b})$

  $(\vec{c}\times\vec{a})\,.\,\vec{b}=-(\vec{a}\times\vec{c})\,.\,\vec{b}=-\vec{b}\,.\,(\vec{a}\times\vec{c})$

- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

  then $(\vec{a}\times\vec{b})\,.\,\vec{c}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

- Four points A, B, C and D are coplanar if $\overrightarrow{AB},\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar i.e. $(\overrightarrow{AB}\times\overrightarrow{AC})\,.\,\overrightarrow{AD}=0$

**Example 34.30** Find the volume of the parallelepiped whose edges are represented by $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$.

**Solution :** Volume $=(\vec{a}\times\vec{b})\,.\,\vec{c}=\begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$

$$= 2(4-1)+3(2+3)+4(-1-6)$$

$$= 6+15-28=-7$$

Neglecting negative sign, required volume = 7 unit$^3$.

**Example 34.31** Find the value of $\lambda$ so that the vectors

$$\vec{a} = 2\hat{i} - \hat{j} + \hat{k},\ b = \hat{i} + 2\hat{j} - 3\hat{k},\ c = 3\hat{i} + \lambda\hat{j} + 5\hat{k}$$ are coplanar.

**Solution :** The vectors $\vec{a}, \vec{b}$ and $\vec{c}$ will be coplanar if $[\vec{a}\vec{b}\vec{c}] = 0$

i.e. $$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0$$

i.e. $2(10 + 3\lambda) + 1(5 + 9) + 1(\lambda - 6) = 0$

i.e. $7\lambda + 28 = 0$

$\Rightarrow$ $\lambda = -4.$

**Example 34.32** Show that the four points A, B, C and D whose position vectors are $(4\hat{i} + 5\hat{j} + \hat{k}), (-\hat{j} - \hat{k}),\ (3\hat{i} + 9\hat{j} + 4\hat{k})$ and $(-4\hat{i} + 4\hat{j} + 4\hat{k})$ respectively are coplanar.

**Solution :**
$$\overrightarrow{AB} = -4\hat{i} - 6\hat{j} - 2\hat{k}$$
$$\overrightarrow{AC} = -\hat{i} + 4\hat{j} + 3\hat{k}$$
$$\overrightarrow{AD} = -8\hat{i} - \hat{j} + 3\hat{k}$$

Now $$(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -4(12 + 3) + 6(-3 + 24) - 2(1 + 32)$$

$$= -60 + 126 - 66 = 0$$

Hence, A, B, C and D are coplanar.

**Example 34.33** Prove that $[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] = 2[\vec{a}, \vec{b}, \vec{c}]$

**Solution :** LHS = $(\vec{a} + \vec{b}) \cdot [(\vec{b} + \vec{c}) \times \vec{c} + \vec{a})]$

$= (\vec{a} + \vec{b}) \cdot [(\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}]$

$= (\vec{a} + \vec{b}) \cdot [(\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}]$ $\quad \because\ \vec{c} \times \vec{c} = 0$

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{a}) + \vec{a} \cdot (\vec{c} \times \vec{a})$

$+ \vec{b} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{a}) + \vec{b} \cdot (\vec{c} \times \vec{a})$

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{c} \times \vec{a})$ [$\because$ scalar triple product is zero when two vector are equal]

$= 2[\vec{a}\ \vec{b}\ \vec{c}]$

$=$ RHS

## CHECK YOUR PROGRESS 34.8

1. Find the volume of the parallelopiped whose edges are represented by $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}, \vec{c} = 3\hat{i} + 2\hat{j} + 5\hat{k}$.

2. Find the value of $\lambda$ so that the vectors $\vec{a} = -4\hat{i} - 6\hat{j} + \lambda\hat{k}, \vec{b} = -\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{c} = -8\hat{i} - \hat{j} + 3\hat{k}$ are coplanar.

## LET US SUM UP

- A physical quantity which can be represented by a number only is called a scalar.
- A quantity which has both magnitude and direction is called a vector.
- A vector whose magnitude is 'a' and direction from A to B can be represented by $\overrightarrow{AB}$ and its magnitude is denoted by $|\overrightarrow{AB}| = a$.
- A vector whose magnitude is equal to the magnitude of another vector $\vec{a}$ but of opposite direction is called negative of the given vector and is denoted by $-\vec{a}$.
- A unit vector is of magnitude unity. Thus, a unit vector parallel to $\vec{a}$ is denoted by $\hat{a}$ and is equal to $\dfrac{\vec{a}}{|\vec{a}|}$.
- A zero vector, denoted by $\vec{0}$, is of magnitude 0 while it has no definite direction.
- Unlike addition of scalars, vectors are added in accordance with triangle law of addition of vectors and therefore, the magnitude of sum of two vectors is always less than or equal to sum of their magnitudes.
- Two or more vectors are said to be collinear if their supports are the same or parallel.
- Three or more vectors are said to be coplanar if their supports are parallel to the same plane or lie on the same plane.
- If $\vec{a}$ is a vector and x is a scalar, then $x\vec{a}$ is a vector whose magnitude is $|x|$ times the magnitude of $\vec{a}$ and whose direction is the same or opposite to that of $\vec{a}$ depending upon $x > 0$ or $x < 0$.
- Any vector co-planar with two given non-collinear vectors is expressible as their linear combination.
- Any vector in space is expressible as a linear combination of three given non-coplanar vectors.
- The position vector of a point that divides the line segment joining the points with position vectors $\vec{a}$ and $\vec{b}$ in the ratio of m : n internally/externally are given by
- $\dfrac{n\vec{a} + m\vec{b}}{m + n}, \dfrac{n\vec{a} - m\vec{b}}{n - m}$ respectively.

- The position vector of mid-point of the line segment joining the points with position vectors $\vec{a}$ and $\vec{b}$ is given by

$$\frac{\vec{a}+\vec{b}}{2}$$

- The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where $\theta$ is the angle between $\vec{a}$ and $\vec{b}$.
- The vector product of two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\theta$ is the angle between $\vec{a}$, $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.
- Direction cosines of a vector are the cosines of the angles subtended by the vector with the positive directions of $x$, $y$ and $z$ axes respectively.
- Any three real numbers which are proportional to the direction cosines of a vector are called direction ratios of that vector.
- Usually, direction cosines of a vector are denoted by $l$, $m$, $n$ and direction ratios by $a$, $b$, $c$.
- $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$, in general.
- If $\overrightarrow{AB} = x\hat{i} + y\hat{j} + z\hat{k}$, then direction ratios of $\overrightarrow{AB}$ are x, y, z and direction cosines are

$$\frac{\pm x}{\sqrt{x^2+y^2+z^2}}, \frac{\pm y}{\sqrt{x^2+y^2+z^2}}, \frac{\pm z}{\sqrt{x^2+y^2+z^2}}.$$

- Direction cosines of a vector are unique but direction ratios are infinite.
- Cross product of two non zero vectors $\vec{a}$ and $\vec{b}$ is defined as $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta.\hat{n}$ where $\theta$ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.
- $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$.
- $\vec{a}\times\vec{b} = \vec{0}$ if either $\vec{a} = 0$ or $\vec{b} = 0$ or $\vec{a}$ and $\vec{b}$ are parallel or $\vec{a}$ and $\vec{b}$ are collinear.
- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$
- $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \vec{j}$.
- $\hat{j}\times\hat{i} = -\hat{k},\ \hat{k}\times\hat{j} = -\hat{i},\ \hat{i}\times\hat{k} = -\vec{j}$.
- $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$
- $\lambda(\vec{a}\times\vec{b}) = (\lambda\vec{a})\times\vec{b} = \vec{a}\times(\lambda\vec{b})$

- $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

- Area of $\Delta = \dfrac{1}{2}|\vec{a}\times\vec{b}|$ where $\vec{a}$ and $\vec{b}$ represent adjacent sides of a triangle.

- Area of || gm = $|\vec{a}\times\vec{b}|$ where $\vec{a}$ and $\vec{b}$ represent adjacent sides of the parallelogram.

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is given by $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}\times\vec{b}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

- If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors then $(\vec{a}\times\vec{b}).\vec{c}$ is called scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$. It is usually denoted as $[\vec{a}\,\vec{b}\,\vec{c}]$

- Volume of parallelepiped $= (\vec{a}\times\vec{b}).\vec{c}$ where $\vec{a}, \vec{b}, \vec{c}$ represent coterminous edges of the parallelopiped.

- $(\vec{a}\times\vec{b}).\vec{c} = 0$, if $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar or any two of the three vectors are equal or parallel.

- $(\vec{a}\times\vec{b}).\vec{c} = \vec{a}.(\vec{b}\times\vec{c})$

- $(\vec{a}\times\vec{b}).\vec{c} = (\vec{b}\times\vec{c}).\vec{a} = (\vec{c}\times\vec{a}).\vec{b}$

- $(\vec{a}\times\vec{b}).\vec{c} = -(\vec{b}\times\vec{a}).\vec{c}$

- Four points A, B, C and D are coplanar if $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar i.e. $(\overrightarrow{AB}\times\overrightarrow{AC}).\overrightarrow{AD} = 0.$

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

$$(\vec{a}\times\vec{b}).\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

## SUPPORTIVE WEB SITES

www.youtube.com/watch?v=ihNZlp7iUHE

http://emweb.unl.edu/math/mathweb/vectors/vectors.html

http://www.mathtutor.ac.uk/geometry_vectors

www.khanacademy.org/.../introduction-to-**vectors-and-scalars**

## TERMINAL EXERCISE

1. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors such that any two of them are non-collinear. Find their sum if the vector $\vec{a}+\vec{b}$ is collinear with the vector $\vec{c}$ and if the vector $\vec{b}+\vec{c}$ is collinear with $\vec{a}$.

2. Prove that any two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exist numbers x and y, both not zero simultaneously, such that $x\vec{a}+y\vec{b}=\vec{0}$.

3. ABCD is a parallelogram in which M is the mid-point of side CD. Express the vectors $\overrightarrow{BD}$ and $\overrightarrow{AM}$ in terms of vectors $\overrightarrow{BM}$ and $\overrightarrow{MC}$.

4. Can the length of the vector $\vec{a}-\vec{b}$ be (i) less than, (ii) equal to or (iii) larger than the sum of the lengths of vectors $\vec{a}$ and $\vec{b}$ ?

5. Let $\vec{a}$ and $\vec{b}$ be two non-collinear vectors. Find the number x and y, if the vector $(2-x)\vec{a}+\vec{b}$ and $y\vec{a}+(x-3)\vec{b}$ are equal.

6. The vectors $\vec{a}$ and $\vec{b}$ are non-collinear. Find the number x if the vector $3\vec{a}+x\vec{b}$ and $(1-x)\vec{a}-\frac{2}{3}\vec{b}$ are parallel.

7. Determine x and y such that the vector $\vec{a}=-2\hat{i}+3\hat{j}+y\hat{k}$ is collinear with the vector $\vec{b}=x\hat{i}-6\hat{j}+2\hat{k}$. Find also the magnitudes of $\vec{a}$ and $\vec{b}$.

8. Determine the magnitudes of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ if $\vec{a}=3\hat{i}-5\hat{j}+8\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}-4\hat{k}$.

9. Find a unit vector in the direction of $\vec{a}$ where $\vec{a}=-6\hat{i}+3\hat{j}-2\hat{k}$.

10. Find a unit vector parallel to the resultant of vectors $3\hat{i}-2\hat{j}+\hat{k}$ and $-2\hat{i}+4\hat{j}+\hat{k}$

11. The following forces act on a particle P :

    $\vec{F}_1=2\hat{i}+\hat{j}-3\hat{k}$, $\vec{F}_2=-3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{F}_3=3\hat{i}-2\hat{j}+\hat{k}$ measured in Newtons.

    Find (a) the resultant of the forces, (b) the magnitude of the resultant.

12. Show that the following vectors are co-planar :

$(\vec{a} - 2\vec{b} + \vec{c}),\ (2\vec{a} + \vec{b} - 3\vec{c})$ and $(-3\vec{a} + \vec{b} + 2\vec{c})$

where $\vec{a}, \vec{b}$ and $\vec{c}$ are any three non-coplanar vectors.

13. A vector makes angles $\frac{\pi}{3}, \frac{\pi}{3}$ with $\overline{OX}$ and $\overline{OY}$ respectively. Find the angle made by it with $\overline{OZ}$.

14. If $P(\sqrt{3}, 1, 2\sqrt{3})$ is a point in space, find direction cosines of $\overline{OP}$ where O is the origin.

15. Find the direction cosines of the vector joining the points $(-4, 1, 7)$ and (2, –3, 2).

16. Using the concept of direction ratios show that $\overrightarrow{PQ} \parallel \overrightarrow{RS}$ where coordinates of P, Q, R and S are (0, 1, 2), (3, 4, 8), $\left(-2, \frac{3}{2}, -3\right)$ and $\left(\frac{5}{2}, 6, 6\right)$ respectively.

17. If the direction ratios of a vector are (3, 4, 0). Find its directions cosines.

18. Find the area of the parallelogram whose adjacent sides are represented by the vectors $\hat{i} + 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

19. Find the area of the $\Delta$ABC where coordinates of A, B, C are (3, –1, 2), (1, –1, –3) and (4, –3, 1) respectively.

20. Find a unit vector perpendicular to each of the vectors $2\hat{i} - 3\hat{j} + \hat{k}$ and $3\hat{i} - 4\hat{j} - \hat{k}$.

21. If $\vec{A} = 2\hat{i} - 3\hat{j} - 6\hat{k}$ and $\vec{B} = \hat{i} + 4\hat{j} - 2\hat{k}$, then find $(\vec{A} + \vec{B}) \times (\vec{A} - \vec{B})$.

22. Prove that : $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.

23. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, show that $(\vec{a} - \vec{d})$ is parallel to $(\vec{b} - \vec{c})$.

24. Find the volume of the parallelepiped whose edges are represented by $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}, \vec{c} = 3\hat{i} - 5\hat{j} + 2\hat{k}$.

25. Show that the vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$ and $\vec{c} = 3\hat{i} - 4\hat{j} - 4\hat{k}$ are coplanar.

26. Find the value of $\lambda$ if the points A(3, 2, 1), B(4, $\lambda$, 5), C(4, 2, –2) and D(6, 5, –1) are coplanar.

MODULE - IX
Vectors and three dimensional Geometry

Notes

## ANSWERS

## CHECK YOUR PROGRESS 34.1

1. (d) 2. (b)

3.

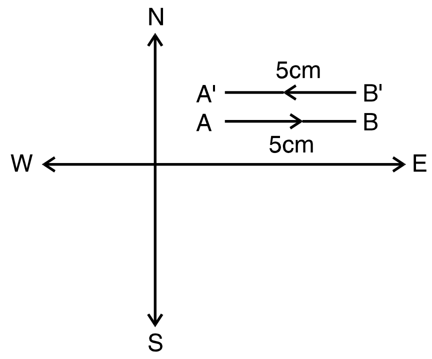



Fig. 34.32

4. Two vectors are said to be like if they have same direction what ever be their magnitudes.

   But in case of equal vectors magnitudes and directions both must be same.

5.

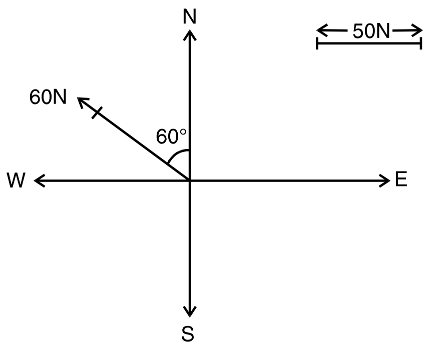



Fig. 34.33

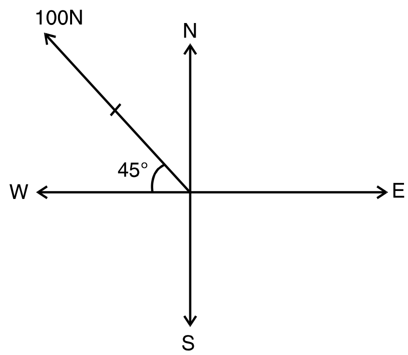



Fig. 34.34

## CHECK YOUR PROGRESS 34.2

1. $\vec{0}$ 2. $\vec{0}$

## CHECK YOUR PROGRESS 34.3

1. $\vec{b} - \vec{a}$

2. (i) It is a vector in the direction of $\vec{a}$ and whose magnitudes is 3 times that of $\vec{a}$.

   (ii) It is a vector in the direction opposite to that of $\vec{b}$ and with magnitude 5 times that of $\vec{b}$.

3. $\overrightarrow{DB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = 2\vec{a} + 3\vec{b}$.

4. $|\, y\vec{n}\,| = y\,|\,\vec{n}\,|$ if $y > 0$

5. Vector $= -y\,|\,\vec{n}\,|$ if $y < 0 = 0$ if $y = 0$

6. $\vec{p} = x\vec{q}$, x is a non-zero scalar.

## CHECK YOUR PROGRESS 34.4

1. If there exist scalars x and y such that $\vec{c} = x\vec{a} + y\vec{b}$

2. $\vec{r} = 3\hat{i} + 4\hat{j}$ 3. $\overrightarrow{OP} = 4\hat{i} + 3\hat{j} + 5\hat{k}$

5. $\frac{1}{7}\left(3\hat{i} + 6\hat{j} - 2\hat{k}\right)$ 6. $\frac{1}{\sqrt{51}}\hat{i} - \frac{5}{\sqrt{51}}\hat{j} - \frac{5}{\sqrt{51}}\hat{k}$

## CHECK YOUR PROGRESS 34.5

1. (i) $\frac{1}{5}(2\vec{a} + 3\vec{b})$ (ii) $(3\vec{a} - 2\vec{b})$

2. $\frac{1}{7}(4\vec{p} + 3\vec{q})$ 3. $\frac{1}{3}(2\vec{c} + \vec{d}),\ \frac{1}{3}(\vec{c} + 2\vec{d})$

## CHECK YOUR PROGRESS 34.6

1. (i) (0, 1, 0) (ii) 1 (iii) not equal (iv) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

(v) proportional (vi) sum of their squares is not equal to 1 (vii) infinite

2. $\frac{3}{5\sqrt{2}}, \frac{2\sqrt{2}}{5}, \frac{-1}{\sqrt{2}}$ 3. $\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$ 4. 0, –1, 1

## CHECK YOUR PROGRESS 34.7

1. (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$ (iii) 1

2. $\frac{-\hat{i}}{\sqrt{6}} + \frac{2\hat{j}}{\sqrt{6}} - \frac{\hat{k}}{\sqrt{6}}$. 3. $\sqrt{42}$ unit$^2$

## CHECK YOUR PROGRESS 34.8

1. 42 unit$^3$ 2. $\lambda = -2$

## TERMINAL EXERCISE

1. $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

3. $\overrightarrow{BD} = \overrightarrow{BM} - \overrightarrow{MC},\ \overrightarrow{AM} = \overrightarrow{BM} + 2\overrightarrow{MC}$

4. (i) Yes, $\vec{a}$ and $\vec{b}$ are either any non-collinear vectors or non-zero vectors of same direction.

   (ii) Yes, $\vec{a}$ and $\vec{b}$ are either in the opposite directions or at least one of them is a zero vector.

   (iii) Yes, $\vec{a}$ and $\vec{b}$ have opposite directions.

5. $x = 4,\ y = -2$ 6. $x = 2,\ -1$

7. $x = 4,\ y = -1$ $|\vec{a}| = \sqrt{14},\ |\vec{b}| = 2\sqrt{14}$

8. $|\vec{a} + \vec{b}| = 6,\ |\vec{a} - \vec{b}| = 14$

9. $-\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{2}{7}\hat{k}$ 10. $\pm\frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

11. $2\hat{i} + \hat{j};\ \sqrt{5}$ 13. $\frac{\pi}{4}$ or $\frac{3\pi}{4}$

14. $\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}$ 15. $\frac{6}{\sqrt{77}}, \frac{-4}{\sqrt{77}}, \frac{-5}{\sqrt{77}}$

17. $\frac{3}{5}, \frac{4}{5}, 0$ 18. $8\sqrt{3}$ unit$^2$

19. $\frac{1}{2}\sqrt{165}$ unit$^2$ 20. $\frac{7\hat{i} + 5\hat{j} + \hat{k}}{\sqrt{75}}$

21. $-60\hat{i} + 4\hat{j} - 22\hat{k}$ 24. 8 unit$^3$

26. $\lambda = 5$