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DIFFERENTIAL EQUATIONS

Having studied the concept of differentiation and integration, we are now faced with the question where do they find an application.

In fact these are the tools which help us to determine the exact takeoff speed, angle of launch, amount of thrust to be provided and other related technicalities in space launches.

Not only this but also in some problems in Physics and Bio-Sciences, we come across relations which involve derivatives.

One such relation could be $\frac{ds}{dt} = 4.9\ t^2$ where *s* is distance and *t* is time. Therefore, $\frac{ds}{dt}$ represents velocity (rate of change of distance) at time *t*.

Equations which involve derivatives as their terms are called differential equations. In this lesson, we are going to learn how to find the solutions and applications of such equations.

OBJECTIVES

After studying this lesson, you will be able to :

- define a differential equation, its order and degree;
- determine the order and degree of a differential equation;
- form differential equation from a given situation;
- illustrate the terms "general solution" and "particular solution" of a differential equation through examples;
- solve differential equations of the following types :

  (i) $\frac{dy}{dx} = f(x)$ (ii) $\frac{dy}{dx} = f(x)\,g(y)$

  (iii) $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ (iv) $\frac{dy}{dx} + P(x)\,y = Q(x)$

- find the particular solution of a given differential equation for given conditions.

EXPECTED BACKGROUND KNOWLEDGE

- Integration of algebraic functions, rational functions and trigonometric functions

32.1 DIFFERENTIAL EQUATIONS

As stated in the introduction, many important problems in Physics, Biology and Social Sciences, when formulated in mathematical terms, lead to equations that involve derivatives. Equations which involve one or more differential coefficients such as $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ (or differentials) etc. and independent and dependent variables are called differential equations.

For example,

(i) $\frac{dy}{dx} = \cos x$ (ii) $\frac{d^2y}{dx^2} + y = 0$ (iii) $xdx + ydy = 0$

(iv) $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0$ (vi) $y = \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

32.2 ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

Order : It is the order of the highest derivative occurring in the differential equation.

Degree : It is the degree of the highest order derivative in the differential equation.

	Differential Equation	Order	Degree
(i)	$\frac{dy}{dx} = \sin x$	One	One
(ii)	$\left(\frac{dy}{dx}\right)^2 + 3y^2 = 5x$	One	Two
(iii)	$\left(\frac{d^2s}{dt^2}\right)^2 + t^2\left(\frac{ds}{dt}\right)^4 = 0$	Two	Two
(iv)	$\frac{d^3v}{dr^3} + \frac{2}{r}\frac{dv}{dr} = 0$	Three	One
(v)	$\left(\frac{d^4y}{dx^4}\right)^2 + x^3\left(\frac{d^3y}{dx^3}\right)^5 = \sin x$	Four	Two

Example 32.1 Find the order and degree of the differential equation :

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right] = 0$$

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right] = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$$

Hence order of the diferential equation is 2 and the degree of the differential equation is 1.

Note : Degree of a differential equation is defiend if it is a palynomial equation in terms of its derivatives.

32.3 LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS

A differential equation in which the dependent variable and all of its derivatives occur only in the first degree and are not multiplied together is called a **linear differential equation**. A differential equation which is not linear is called non-linear differential equation . For example, the differential equations

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{and} \quad \cos^2 x \frac{d^3y}{dx^3} + x^3 \frac{dy}{dx} + y = 0 \text{ are linear.}$$

The differential equation $\left(\frac{dy}{dx}\right)^2 + \frac{y}{x} = \log x$ is non-linear as degree of $\frac{dy}{dx}$ is two.

Further the differential equation $y \frac{d^2y}{dx^2} - 4 = x$ is non-linear because the dependent variable *y* and its derivative $\frac{d^2y}{dx^2}$ are multiplied together.

32.4 FORMATION OF A DIFFERENTIAL EQUATION

Consider the family of all straight lines passing through the origin (see Fig. 28.1).

This family of lines can be represented by

$$y = mx \qquad .....(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = m \qquad .....(2)$$

From (1) and (2), we get

MODULE - VIII
Calculus

Notes

$$y = x\frac{dy}{dx} \quad .....(3)$$

So $y = mx$ and $y = x\frac{dy}{dx}$ represent the same family.

Clearly equation (3) is a differential equation.

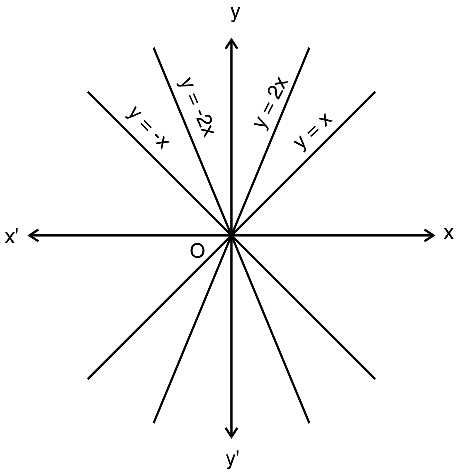


Fig. 32.1

Working Rule : To form the differential equation corresponding to an equation involving two variables, say *x* and *y* and some arbitrary constants, say, *a*, *b*, *c*, etc.

(i) Differentiate the equation as many times as the number of arbitrary constants in the equation.

(ii) Eliminate the arbitrary constants from these equations.

Remark

If an equation contains n arbitrary constants then we will obtain a differential equation of n^{th} order.

Example 32.2 Form the differential equation representing the family of curves.

$$y = ax^2 + bx. \quad ......(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = 2ax + b \quad ......(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = 2a \quad ......(3)$$

$$\Rightarrow \quad a = \frac{1}{2}\frac{d^2y}{dx^2} \quad ......(4)$$

(The equation (1) contains two arbitrary constants. Therefore, we differentiate this equation two times and eliminate 'a' and 'b').

On putting the value of 'a' in equation (2), we get

$$\frac{dy}{dx} = x\frac{d^2y}{dx^2} + b$$

$$\Rightarrow \quad b = \frac{dy}{dx} - x\frac{d^2y}{dx^2} \quad ......(5)$$

Substituting the values of 'a' and 'b' given in (4) and (5) above in equation (1), we get

$$y = x^2\left(\frac{1}{2}\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)$$

or $$y = \frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2}$$

or $$y = x\frac{dy}{dx} - \frac{x^2}{2}\frac{d^2y}{dx^2}$$

or $$\frac{x^2}{2}\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

which is the required differential equation.

Example 32.3 Form the differential equation representing the family of curves

$$y = a\cos(x + b).$$

Solution : $$y = a\cos(x + b) \quad .....(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = -a\sin(x + b) \quad ....(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -a\cos(x + b) \quad .....(3)$$

From (1) and (3), we get

$$\frac{d^2y}{dx^2} = -y \quad \text{or} \quad \frac{d^2y}{dx^2} + y = 0$$

which is the required differential equation.

Example 32.4 Find the differential equation of all circles which pass through the origin and whose centres are on the x-axis.

Solution : As the centre lies on the x-axis, its coordinates will be (a, 0).

Since each circle passes through the origin, its radius is a.

Then the equation of any circle will be

$$(x - a)^2 + y^2 = a^2 \quad (1)$$

To find the corresponding differential equation, we differentiate equation (1) and get

$$2(x - a) + 2y\frac{dy}{dx} = 0$$

or $$x - a + y\frac{dy}{dx} = 0$$

or $$a = y\frac{dy}{dx} + x$$

Substituting the value of 'a' in equation (1), we get

$$\left(x - y\frac{dy}{dx} - x\right)^2 + y^2 = \left(y\frac{dy}{dx} + x\right)^2$$

or $$\left(y\frac{dy}{dx}\right)^2 + y^2 = x^2 + \left(y\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$$

or $$y^2 = x^2 + 2xy\frac{dy}{dx}$$

which is the required differential equation.

Remark

If an equation contains one arbitrary constant then the corresponding differential equation is of the first order and if an equation contains two arbitrary constants then the corresponding differential equation is of the second order and so on.

or $$\frac{dr}{dt} = k$$

which is the required differential equation.

CHECK YOUR PROGRESS 32.1

1. Find the order and degree of the differential equation

$$y = x\frac{dy}{dx} + 1$$

2. Write the order and degree of each of the following differential equations.

(a) $$\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$$

(b) $$\left(\frac{d^2s}{dt^2}\right)^2 + 3\left(\frac{ds}{dt}\right)^3 + 4 = 0$$

3. State whether the following differential equations are linear or non-linear.

(a) $\left(xy^2 - x\right)dx + \left(y - x^2y\right)dy = 0$ (b) $dx + dy = 0$

(c) $\frac{dy}{dx} = \cos x$ (d) $\frac{dy}{dx} + \sin^2 y = 0$

4. Form the differential equation corresponding to

$(x - a)^2 + (y - b)^2 = r^2$ by eliminating 'a' and 'b'.

5. (a) Form the differential equation corresponding to

$y^2 = m\left(a^2 - x^2\right)$

(b) Form the differential equation corresponding to

$y^2 - 2ay + x^2 = a^2$, where a is an arbitrary constant.

(c) Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-3x}$ where A and B are arbitrary constants.

(d) Find the differential equation of all straight lines passing through the point (3,2).

(e) Find the differential equation of all the circles which pass through origin and whose centres lie on y-axis.

32.5 GENERAL AND PARTICULAR SOLUTIONS

Finding solution of a differential equation is a reverse process. Here we try to find an equation which gives rise to the given differential equation through the process of differentiations and elimination of constants. The equation so found is called the primitive or the solution of the differential equation.

Remarks

(1) If we differentiate the primitive, we get the differential equation and if we integrate the differential equation, we get the primitive.

(2) Solution of a differential equation is one which satisfies the differential equation.

Example 32.5 Show that $y = C_1 \sin x + C_2 \cos x$, where C_1 and C_2 are arbitrary constants, is a solution of the differential equation :

$$\frac{d^2y}{dx^2} + y = 0$$

Solution : We are given that

$$y = C_1 \sin x + C_2 \cos x \qquad .....(1)$$

Differentiating both sides of (1), we get

$$\frac{dy}{dx} = C_1 \cos x - C_2 \sin x \qquad .....(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -C_1 \sin x - C_2 \cos x$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation, we get

$$\frac{d^2y}{dx^2} + y = C_1 \sin x + C_2 \cos x + (-C_1 \sin x - C_2 \cos x)$$

or $$\frac{d^2y}{dx^2} + y = 0$$

In integration, the arbitrary constants play important role. For different values of the constants we get the different solutions of the differential equation.

A solution which contains as many as arbitrary constants as the order of the differential equation is called the **General Solution** or complete primitive.

If we give the particular values to the arbitrary constants in the general solution of differential equation, the resulting solution is called a **Particular Solution**.

Remark

General Solution contains as many arbitrary constants as is the order of the differential equation.

Example 32.6 Show that $y = cx + \frac{a}{c}$ (where c is a constant) is a solution of the differential equation.

$$y = x\frac{dy}{dx} + a\frac{dx}{dy}$$

Solution : We have $y = cx + \frac{a}{c}$ .....(1)

Differentiating (1), we get

$$\frac{dy}{dx} = c \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1}{c}$$

On substituting the values of $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in R.H.S of the differential equation, we have

$$x(c) + a\left(\frac{1}{c}\right) = cx + \frac{a}{c} = y$$

$\Rightarrow$ R.H.S. = L.H.S.

Hence $y = cx + \frac{a}{c}$ is a solution of the given differential equation.

Example 32.7 If $y = 3x^2 + C$ is the general solution of the differential equation $\frac{dy}{dx} - 6x = 0$, then find the particular solution when y = 3, x = 2.

Solution : The general solution of the given differential equation is given as

$$y = 3x^2 + C \qquad ....(1)$$

Now on substituting y = 3, x = 2 in the above equation , we get

$$3 = 12 + C \qquad \text{or} \qquad C = -9$$

By substituting the value of C in the general solution (1), we get

$$y = 3x^2 - 9$$

which is the required particular solution.

32.6 TECHNIQUES OF SOLVING IN G A DIFFERENTIAL EQUATION

32.6.1 When Variables are Separable

(i) Differential equation of the type $\frac{dy}{dx} = f(x)$

Consider the differential equation of the type $\frac{dy}{dx} = f(x)$

or $\qquad dy = f(x)\, dx$

On integrating both sides, we get

$$\int dy = \int f(x)\, dx$$

$$y = \int f(x)\, dx + c$$

where c is an arbitrary constant. This is the general solution.

Note : It is necessary to write c in the general solution, otherwise it will become a particular solution.

Example 32.8 Solve

$$(x + 2)\frac{dy}{dx} = x^2 + 4x - 5$$

Solution : The given differential equation is $(x + 2)\frac{dy}{dx} = x^2 + 4x - 5$

or $\dfrac{dy}{dx} = \dfrac{x^2 + 4x - 5}{x + 2}$ or $\dfrac{dy}{dx} = \dfrac{x^2 + 4x + 4 - 4 - 5}{x + 2}$

or $\dfrac{dy}{dx} = \dfrac{(x + 2)^2}{x + 2} - \dfrac{9}{x + 2}$ or $\dfrac{dy}{dx} = x + 2 - \dfrac{9}{x + 2}$

or $$dy = \left(x + 2 - \frac{9}{x + 2} \right) dx \qquad .....(1)$$

On integrating both sides of (1), we have

$$\int dy = \int \left(x + 2 - \frac{9}{x + 2} \right) dx \quad \text{or} \quad y = \frac{x^2}{2} + 2x - 9 \log |x + 2| + c,$$

where c is an arbitrary constant, is the required general solution.

Example 32.9 Solve

$$\frac{dy}{dx} = 2x^3 - x$$

given that y = 1 when x = 0

Solution : The given differential equation is $\dfrac{dy}{dx} = 2x^3 - x$

or $$dy = \left(2x^3 - x \right) dx \qquad ...(1)$$

On integrating both sides of (1), we get

$$\int dy = \int \left(2x^3 - x \right) dx \quad \text{or} \quad y = 2.\frac{x^4}{4} - \frac{x^2}{2} + C$$

or $$y = \frac{x^4}{2} - \frac{x^2}{2} + C \qquad .....(2)$$

where C is an arbitrary constant.

Since y = 1 when x = 0, therefore, if we substitute these values in (2) we will get

$$1 = 0 - 0 + C \quad \Rightarrow \quad C = 1$$

Now, on putting the value of C in (2), we get

$$y = \frac{1}{2}\left(x^4 - x^2 \right) + 1 \quad \text{or} \quad y = \frac{1}{2} x^2 \left(x^2 - 1 \right) + 1$$

which is the required particular solution.

(ii) Differential equations of the type $\dfrac{dy}{dx} = f(x) \cdot g(y)$

Consider the differential equation of the type

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

or $$\frac{dy}{g(y)} = f\ (x)\ dx \qquad .....(1)$$

In equation (1), x's and y's have been separated from one another. Therefore, this equation is also known differential equation with variables separable.

To solve such differential equations, we integrate both sides and add an arbitrary constant on one side.

To illustrate this method, let us take few examples.

Example 32.10 Solve

$$\left(1+x^2\right)dy = \left(1+y^2\right)dx$$

Solution : The given differential equation

$$\left(1+x^2\right)dy = \left(1+y^2\right)dx$$

can be written as $$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$ (Here variables have been seperated)

On integrating both sides of (1), we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

or $$\tan^{-1} y = \tan^{-1} x + C$$

where C is an arbitrary constant.

This is the required solution.

Example 32.11 Find the particular solution of

$$\frac{dy}{dx} = \frac{2x}{3y^2+1}$$

when y (0) = 3 (i.e. when x = 0, y = 3).

Solution : The given differential equation is

$$\frac{dy}{dx} = \frac{2x}{3y^2+1} \text{ or } \quad \left(3y^2+1\right)dy = 2x\,dx \text{ (Variables separated)} \qquad ....(1)$$

If we integrate both sides of (1), we get

$$\int\left(3y^2+1\right)dy = \int 2x\,dx,$$

where C is an arbitrary constant.

$$y^3 + y = x^2 + C \qquad ....(2)$$

It is given that, $y(0) = 3$.

$\therefore$ on substituting y = 3 and x = 0 in (2), we get

$$27 + 3 = C$$

$$\therefore \quad C = 30$$

Thus, the required particular solution is

$$y^3 + y = x^2 + 30$$

32.6.2 Homogeneous Differential Equations

Consider the following differential equations :

(i) $y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$ (ii) $\left(x^3 + y^3\right)dx - 3xy^2dy = 0$

(iii) $\frac{dy}{dx} = \frac{x^3 + xy^2}{y^2x}$

In equation (i) above, we see that each term except $\frac{dy}{dx}$ is of degree 2

[as degree of y^2 is 2, degree of x^2 is 2 and degree of xy is 1 + 1 =2]

In equation (ii) each term except $\frac{dy}{dx}$ is of degree 3.

In equation (iii) each term except $\frac{dy}{dx}$ is of degree 3, as it can be rewritten as

$$y^2x \frac{dy}{dx} = x^3 + x\,y^2$$

Such equations are called **homogeneous equations**.

Remarks

Homogeneous equations do not have constant terms.

For example, differential equation

$$\left(x^2 + 3yx\right)dx - \left(x^3 + x\right)dy = 0$$

is not a homogeneous equation as the degree of the function except $\frac{dy}{dx}$ in each term is not the same. [degree of x^2 is 2, that of 3yx is 2, of x^3 is 3, and of x is 1]

32.6.3 Solution of Homogeneous Differential Equation :

To solve such equations, we proceed in the following manner :

(i) write one variable = v. (the other variable).
(i.e. either y = vx or x = vy)

(ii) reduce the equation to separable form

(iii) solve the equation as we had done earlier.

Example 32.12 Solve

$$\left(x^2 + 3xy + y^2\right)dx - x^2dy = 0$$

Solution : The given differential equation is

$$\left(x^2 + 3xy + y^2\right)dx - x^2dy = 0$$

or $$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2} \qquad .....(1)$$

It is a homogeneous equation of degree two. (Why?)

Let $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ From (1), we have

$$v + x\frac{dv}{dx} = \frac{x^2 + 3x.vx + (vx)^2}{x^2} \quad \text{or} \quad v + x\frac{dv}{dx} = x^2\left[\frac{1 + 3v + v^2}{x^2}\right]$$

or $$v + x\frac{dv}{dx} = 1 + 3v + v^2 \quad \text{or} \quad x\frac{dv}{dx} = 1 + 3v + v^2 - v$$

or $$x\frac{dv}{dx} = v^2 + 2v + 1 \quad \text{or} \quad \frac{dv}{v^2 + 2v + 1} = \frac{dx}{x}$$

or $$\frac{dv}{(v+1)^2} = \frac{dx}{x} \qquad .....(2)$$

Further on integrating both sides of (2), we get

$$\frac{-1}{v+1} + C = \log|x|,$$ where C is an arbitrary constant.

On substituting the value of v, we get

$$\frac{x}{y+x} + \log|x| = C$$ which is the required solution.

Note: If the Homoqeneous differential equation is written in the form $\frac{dx}{qy} = \frac{P(x,y)}{Q(x,y)}$ then x=vy is substituted to find solution.

32.6.4 Differential Equation of the type $\frac{dy}{qx} + py = Q$, where P and Q are functions of x only.

Consider the equation

$$\frac{dy}{dx} + Py = Q \qquad .....(1)$$

where P and Q are functions of x. This is linear equation of order one.

To solve equation (1), we first multiply both sides of equation (1) by $e^{\int Pdx}$ (called integrating factor) and get

$$e^{\int Pdx}\frac{dy}{dx} + Py\, e^{\int Pdx} = Qe^{\int Pdx}$$

or $$\frac{d}{dx}\left(ye^{\int Pdx}\right) = Qe^{\int Pdx} \qquad .....(2)$$

$$\left[\because \frac{d}{dx}\left(ye^{\int Pdx}\right) = e^{\int Pdx}\frac{dy}{dx} + Py.e^{\int Pdx}\right]$$

On integrating, we get

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C \qquad .....(3)$$

where C is an arbitrary constant,

or $$y = e^{-\int Pdx}\left[\int Qe^{\int Pdx}dx + C\right]$$

Note : $e^{\int Pdx}$ is called the integrating factor of the equation and is written as I.F in short.

Remarks

(i) We observe that the left hand side of the linear differential equation (1) has become $\frac{d}{dx}\left(ye^{\int Pdx}\right)$ after the equation has been multiplied by the factor $e^{\int Pdx}$.

(ii) The solution of the linear differential equation

$$\frac{dy}{dx} + Py = Q$$

P and Q being functions of x only is given by

$$ye^{\int Pdx} = \int Q\left(e^{\int Pdx}\right)dx + C$$

(iii) The coefficient of $\frac{dy}{dx}$, if not unity, must be made unity by dividing the equation by it throughout.

(iv) Some differential equations become linear differential equations if y is treated as the independent variable and x is treated as the dependent variable.

For example, $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y only, is also a linear differential equation of the first order.

In this case $\quad \text{I.F.} = e^{\int P dy}$

and the solution is given by

$$x(\text{I.F.}) = \int Q.(\text{I.F.})dy + C$$

Example 32.13 Solve

$$\frac{dy}{dx} + \frac{y}{x} = e^{-x}$$

Solution : Here $P = \frac{1}{x}, Q = e^{-x}$ (Note that both P an Q are functions of x)

I.F. (Integrating Factor) $e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x \qquad (x > 0)$

$\therefore$ Solution of the given equation is:

$$yx = \int xe^{-x} dx + C$$

where C is an arbitrary constant

or $\quad xy = -xe^{-x} + \int e^{-x} dx + C$

or $\quad xy = -xe^{-x} - e^{-x} + C$

or $\quad xy = -e^{-x}(x+1) + C$

or $\quad y = -\left(\frac{x+1}{x}\right)e^{-x} + \frac{C}{x}$

Note: In the solution x > 0.

Example 32.14 Solve :

$$\sin x \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$$

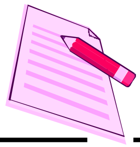
Solution : The given differential equation is

$$\sin x \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$$

or $\quad \frac{dy}{dx} + y \cot x = 2 \sin x \cos x \qquad .....(1)$

Here $\quad P = \cot x, Q = 2 \sin x \cos x$

$$\text{I.F.} = e^{\int P dx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$$



$\therefore$ Solution of the given equation is:

$$y \sin x = \int 2\sin^2 x \cos x \, dx + C$$

where C is an arbitrary constant (sin x >0)

or $$y \sin x = \frac{2}{3}\sin^3 x + C,$$ which is the required solution.

Example 32.15 Solve $\left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$

Solution : The given differential equation is

$$\left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$$

or $$\frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^2} - \frac{x}{1+y^2}$$

or $$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \qquad ....(1)$$

which is of the form $\frac{dx}{dy} + Px = Q$, where P and Q are the functions of y only.

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$\therefore$ Solution of the given equation is:

$$x - e^{\tan -1 y} = \left(\frac{e^{\tan -1 y}}{1+y^2}\right) e^{\tan -1 y} dy + c.$$

where C is an arbitrary constant let $t = \tan^{-1} y$ therefore $dt = \frac{1}{1+y^2} dy$

or $$\left(e^{\tan^{-1} y}\right) x = \int e^t . tdt + C,$$

or $$\left(e^{\tan^{-1} y}\right) x = te^t - \int e^t + C$$

or $$\left(e^{\tan^{-1} y}\right) x = te^t - e^t + C$$

or $$\left(e^{\tan^{-1} y}\right) x = \tan^{-1} y \, e^{\tan^{-1} y} - e^{\tan^{-1} y} + C \qquad (\text{on putting } t = \tan^{-1} y)$$

or $$x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$$

CHECK YOUR PROGRESS 32.2

1. (i) Is $y = \sin x$, a solution of $\frac{d^2y}{dx^2} + y = 0$?

   (ii) Is $y = x^3$, a solution of $x\frac{dy}{dx} - 4y = 0$?

2. Given below are some solutions of the differential equation $\frac{dy}{dx} = 3x$.

   State which are particular solutions and which are general solutions.

   (i) $2y = 3x^2$ (ii) $y = \frac{3}{2}x^2 + 2$

   (iii) $2y = 3x^2 + C$ (iv) $y = \frac{3}{2}x^2 + 3$

3. State whether the following differential equations are homogeneous or not ?

   (i) $\frac{dy}{dx} = \frac{x^2}{1+y^2}$ (ii) $\left(3xy + y^2\right)dx + \left(x^2 + xy\right)dy = 0$

   (iii) $(x+2)\frac{dy}{dx} = x^2 + 4x - 9$ (iv) $\left(x^3 - yx^2\right)dy + \left(y^3 + x^3\right)dx = 0$

4. (a) Show that $y = a \sin 2x$ is a solution of $\frac{d^2y}{dx^2} + 4y = 0$

   (b) Verify that $y = x^3 + ax^2 + c$ is a solution of $\frac{d^3y}{dx^3} = 6$

5. The general solution of the differential equation

   $\frac{dy}{dx} = \sec^2 x$ is $y = \tan x + C$.

   Find the particular solution when

   (a) $x = \frac{\pi}{4}, y = 1$ (b) $x = \frac{2\pi}{3}, y = 0$

6. Solve the following differential equations :

   (a) $\frac{dy}{dx} = x^5 \tan^{-1}\left(x^3\right)$ (b) $\frac{dy}{dx} = \sin^3 x \cos^2 x + xe^x$

   (c) $\left(1+x^2\right)\frac{dy}{dx} = x$ (d) $\frac{dy}{dx} = x^2 + \sin 3x$

7. Find the particular solution of the equation $e^x \frac{dy}{dx} = 4$, given that $y = 3$, when $x = 0$

8. Solve the following differential equations :

(a) $\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2 = 0$ (b) $\frac{dy}{dx} = xy + x + y + 1$

(c) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$ (d) $\frac{dy}{dx} = e^{x-y} + e^{-y}x^2$

9. Solve the following differential equations :

(a) $\left(x^2 + y^2\right)dx - 2xy\, dy = 0$ (b) $x\frac{dy}{dx} + \frac{y^2}{x} = y$

(c) $\frac{dy}{dx} = \frac{\sqrt{x^2 - y^2} + y}{x}$ (d) $\frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right)$

10. Solve : $\frac{dy}{dx} + y \sec x = \tan x$

11. Solve the following differential equations :

(a) $\left(1 + x^2\right)\frac{dy}{dx} + y = \tan^{-1} x$ (b) $\cos^2 x \frac{dy}{dx} + y = \tan x$

(c) $x \log x \frac{dy}{dx} + y = 2 \log x,\ x > 1$

12. Solve the following differential equations:

(a) $(x + y + 1)\frac{dy}{dx} = 1$

[Hint: $\frac{dx}{dy} = x + y + 1$ or $\frac{dx}{dy} - x = y + 1$ which is of the form $\frac{dx}{dy} + Px = Q$]

(b) $\left(x + 2y^2\right)\frac{dy}{dx} = y,\ y > 0$ [Hint: $y\frac{dx}{dy} = x + 2y^2$ or $\frac{dx}{dy} - \frac{x}{y} = 2y$]

Example 32.16 Verify if $y = e^{m \sin^{-1} x}$ is a solution of

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - \frac{dy}{dx} - m^2 y = 0$$

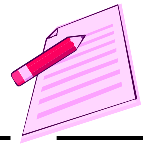
Solution : We have,

$$y = e^{m \sin^{-1} x} \qquad .....(1)$$

Differentiating (1) w.r.t. x , we get

$$\frac{dy}{dx} = \frac{m e^{m \sin^{-1} x}}{\sqrt{1 - x^2}} = \frac{my}{\sqrt{1 - x^2}}$$

or $$\sqrt{1 - x^2}\frac{dy}{dx} = my$$



Squaring both sides, we get

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = m^2y^2$$

Differentiating both sides, we get

$$-2x\left(\frac{dy}{dx}\right)^2 + 2\left(1-x^2\right)\frac{dy}{dx}.\frac{d^2y}{dx^2} = 2m^2y\frac{dy}{dx}$$

or $$-x\frac{dy}{dx} + \left(1-x^2\right)\frac{d^2y}{dx^2} = m^2y$$

or $$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$

Hence $y = e^{m\sin^{-1}x}$ is the solution of

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$

Example 32.17 Find the equation of the curve represented by

$$(y - yx)dx + (x + xy)dy = 0$$

and passing through the point (1, 1).

Solution : The given differential equation is

$$(y - yx)dx + (x + xy)dy = 0$$

or $$(x + xy)dy = (yx - y)dx$$

or $$x(1+y)dy = y(x-1)dx$$

or $$\frac{(1+y)}{y}dy = \frac{x-1}{x}dx \qquad .....(1)$$

Integrating both sides of equation (1), we get

$$\int\left(\frac{1+y}{y}\right)dy = \int\left(\frac{x-1}{x}\right)dx$$

or $$\int\left(\frac{1}{y}+1\right)dy = \int\left(1-\frac{1}{x}\right)dx \qquad .....(2)$$

or $$\log y + y = x - \log x + C$$

Since the curve is passing through the point (1,1), therefore,

substituting x = 1, y = 1 in equation (2), we get

$$1 = 1 + C$$

or $$C = 0$$

Thus, the equation of the required curve is

$$\log y + y = x - \log x$$

or $$\log(xy) = x - y$$

Example 32.18 Solve $\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}$

Solution : We have $\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}$

or $$\frac{dy}{dx} = \frac{3e^{3x}\left(e^{-x} + e^x\right)}{e^x + e^{-x}}$$ or $$\frac{dy}{dx} = 3e^{3x}$$

or $$dy = 3e^{3x}dx \qquad .....(1)$$

Integrating both sides of (1), we get

$$y = \int 3e^{3x}dx + C$$

where C is an arbitrary constant.

or $$y = 3\frac{e^{3x}}{3} + C$$ or $$y = e^{3x} + C$$

which is required solution.

$$y(1 + ax)\left(1 - a^2\right) = x(1 - ay)\left(1 + a^2\right)$$

which is the required solution.

CHECK YOUR PROGRESS 32.3

1. (a) If $y = \tan^{-1} x$, prove that $\left(1 + x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

   (b) $y = e^x \sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

2. (a) Find the equation of the curve represented by

   $\frac{dy}{dx} = xy + x + y + 1$ and passing through the point (2, 0)

   (b) Find the equation of the curve represented by

   $\frac{dy}{dx} + y \cot x = 5e^{\cos x}$ and passing through the point $\left(\frac{\pi}{2}, 2\right)$

3. Solve : $\frac{dy}{dx} = \frac{4e^{3x} + 4e^{5x}}{e^{x} + e^{-x}}$

4. Solve the following differential equations :

(a) $dx + xdy = e^{-y} \sec^2 y \, dy$

(b) $\left(1 + x^2\right)\frac{dy}{dx} - 4x = 3\cot^{-1} x$

(c) $(1 + y)\, xy\, dy = \left(1 - x^2\right)(1 - y)\, dx$

LET US SUM UP

- A differential equation is an equation involving independent variable, dependent variable and the derivatives of dependent variable (and differentials) with respect to independent variable.
- The order of a differential equation is the order of the highest derivative occurring in it.
- The degree of a differential equation is the degree of the highest derivative.
- Degree of a differential equation exists, if it is a polynomial equation in terms of its derivatives.
- A differential equation in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together is called a linear differential equation.
- A linear differential equation is always of the first degree.
- A general solution of a differential equation is that solution which contains as many as the number of arbitrary constants as the order of the differential equation.
- A general solution becomes a particular solution when particular values of the arbitrary constants are determined satisfying the given conditions.
- The solution of the differential equation of the type $\frac{dy}{dx} = f(x)$ is obtained by integrating both sides.
- The solution of the differential equation of the type $\frac{dy}{dx} = f(x)\, g(y)$ is obtained after separating the variables and integrating both sides.
- The differential equation $M(x, y)\, dx + N(x, y)\, dy = 0$ is called homogeneous if $M(x, y)$ and $N(x, y)$ are homogeneous and are of the same degree.
- The solution of a homogeneous differential equation is obtained by substituting $y = vx$ or $x = vy$ and then separating the variables.
- The solution of the first order linear equation $\frac{dy}{dx} + Py = Q$ is

$$ye^{\int P dx} = \int Q\left(e^{\int P dx}\right)dx + C, \qquad \text{where C is an arbitrary constant.}$$

The expression $e^{\int P dx}$ is called the integrating factor of the differential equation and is written as I.F. in short.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=9Wfn-WWV1aY

http://www.youtube.com/watch?v=6YRGEsQWZzY

TERMINAL EXERCISE

1. Find the order and degree of the differential equation :

(a) $\left(\dfrac{d^2y}{dx^2}\right)^2 + x^2\left(\dfrac{dy}{dx}\right)^4 = 0$ (b) $xdx + ydy = 0$

(c) $\dfrac{d^4y}{dx^4} - 4\dfrac{dy}{dx} + 4y = 5\cos 3x$ (d) $\dfrac{dy}{dx} = \cos x$

(e) $x^2\dfrac{d^2y}{dx^2} - xy\dfrac{dy}{dx} = y$ (f) $\dfrac{d^2y}{dx^2} + y = 0$

2. Find which of the following equations are linear and which are non-linear

(a) $\dfrac{dy}{dx} = \cos x$ (b) $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2 \log x$

(c) $\left(\dfrac{d^2y}{dx^2}\right)^3 + x^2\left(\dfrac{dy}{dx}\right)^2 = 0$ (d) $x\dfrac{dy}{dx} - 4 = x$

(e) $dx + dy = 0$

3. Form the differential equation corresponding to $y^2 - 2ay + x^2 = a^2$ by eliminating a.

4. Find the differential equation by eliminating a, b, c from

$y = ax^2 + bx + c$. Write its order and degree.

5. How many constants are contained in the general solution of

(a) Second order differential equation.

(b) Differential equation of order three.

(c) Differential equation of order five.

6. Show that $y = a\cos(\log x) + b\sin(\log x)$ is a solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

7. Solve the following differential equations:

(a) $\sin^2 x \dfrac{dy}{dx} = 3\cos x + 4$ (b) $\dfrac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

(c) $\dfrac{dy}{dx} + \dfrac{\cos x \sin y}{\cos y} = 0$ (d) $dy + xydx = xdx$

(e) $\dfrac{dy}{dx} + y\tan x = x^m \cos mx$ (f) $\left(1 + y^2\right)\dfrac{dx}{dy} = \tan^{-1} y - x$

ANSWERS

CHECK YOUR PROGRESS 32.1

1. Order is 1 and degree is 1.
2. (a) Order 2, degree 1

   (b) Order 2, degree 2
3. (a) Non- linear (b) Linear

   (c) Linear (d) Non-linear
4. $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$
5. (a) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

   (b) $\left(x^2 - 2y^2\right)\left(\frac{dy}{dx}\right)^2 - 4xy\frac{dy}{dx} - x^2 = 0$

   (c) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

   (d) $y = (x-3)\frac{dy}{dx} + 2$

   (e) $\left(x^2 - y^2\right)\frac{dy}{dx} - 2xy = 0$

CHECK YOUR PROGRESS 32.2

1. (i) Yes (ii) No
2. (i), (ii) and (iv) are particular solutions (iii) is the general solution
3. (ii), (iv) are homogeneous
5. (a) $y = \tan x$

   (b) $y = \tan x + \sqrt{3}$
6. (a) $y = \frac{1}{6}x^6 \tan^{-1}\left(x^3\right) - \frac{1}{6}x^3 + \frac{1}{6}\tan^{-1}\left(x^3\right) + C$

   (b) $y = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + (x-1)e^x + C$

(c) $y = \frac{1}{2}\log\left|x^2+1\right| + C$ (d) $y = \frac{1}{3}x^3 - \frac{1}{3}\cos 3x + C$

7. $y = -4e^{-x} + 7$

8. (a) $\log\left|\frac{x}{y}\right| = C + \frac{1}{x} + \frac{1}{y}$ (b) $\log|y+1| = x + \frac{x^2}{2} + C$

(c) $\tan x \tan y = C$ (d) $e^y = e^x + \frac{x^3}{3} + C$

9. (a) $x = C\left(x^2 - y^2\right)$ (b) $x + cy = y\log|x|$

(c) $\sin^{-1}\left(\frac{y}{x}\right) = \log|x| + C$ (d) $\tan\frac{y}{2x} = Cx$

10. $y(\sec x + \tan x) = \sec x + \tan x - x + C$

11. (a) $y = \tan^{-1} x - 1 + Ce^{-\tan x}$ (b) $y = \tan x - 1 + Ce^{-\tan x}$

(c) $y = \log x + \frac{C}{\log x}$

12. (a) $x = Ce^y - (y+2)$ (b) $x = y^2 + Cy$

CHECK YOUR PROGRESS 32.3

2. (a) $\log(y+1) = \frac{1}{2}x^2 + x - 4$ (b) $y\sin x + 5e^{\cos x} = 7$

3. $y = \frac{4}{5}e^{5x} + C$

4. (a) $x = e^{-y}(C + \tan y)$ (b) $y = 2\log\left|1+x^2\right| - \frac{3}{2}\left(\cot^{-1} x\right)^2 + C$

(c) $\log x + 2\log|1-y| = \frac{x^2}{2} - \frac{y^2}{2} - 2y + C$

TERMINAL EXERCISE

1. (a) Order 2, degree 3 (b) Order 1, degree 1

(c) Order 4, degree 1 (d) Order 1, degree 1

(e) Order 2, degree 1 (f) Order 2, degree 1

2. (a), (d), (e) are linear; (b), (c) are non-linear

3. $\left(x^2 - 2y^2\right)\left(\frac{dy}{dx}\right)^2 - 4xy\left(\frac{dy}{dx}\right) - x^2 = 0$

4. $\frac{d^3y}{dx^3} = 0$, Order 3, degree 1.

5. (a) Two (b) Three

(c) Five

7. (a) $y + 3 \operatorname{cosec} x + 4 \cot x = C$ (b) $e^y = e^x + \frac{x^3}{3} + C$

(c) $\sin y = Ce^{-\sin x}$ (d) $\log(1 - y) + \frac{x^2}{2} = C$

(e) $y = \frac{x^{m+1}}{m+1} \cos x + C \cos x$ (f) $x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$