31

311en31

DEFINITE INTEGRALS

In the previous lesson we have discussed the anti-derivative, i.e., integration of a function.The very word integration means to have some sort of summation or combining of results.

Now the question arises : Why do we study this branch of Mathematics? In fact the integration helps to find the areas under various laminas when we have definite limits of it. Further we will see that this branch finds applications in a variety of other problems in Statistics, Physics, Biology, Commerce and many more.

In this lesson, we will define and interpret definite integrals geometrically, evaluate definite integrals using properties and apply definite integrals to find area of a bounded region.

OBJECTIVES

After studying this lesson, you will be able to :

- define and interpret geometrically the definite integral as a limit of sum;
- evaluate a given definite integral using above definition;
- state fundamental theorem of integral calculus;
- state and use the following properties for evaluating definite integrals :

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (ii) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

(iii) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(v) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(vi) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(2a-x) = f(x)$

$= 0 \text{ if } f(2a-x) = -f(x)$

(vii) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if f is an even function of x

$= 0$ if f is an odd function of x.

- apply definite integrals to find the area of a bounded region.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of integration
- Area of a bounded region

31.1 DEFINITE INTEGRAL AS A LIMIT OF SUM

In this section we shall discuss the problem of finding the areas of regions whose boundary is not familiar to us. (See Fig. 31.1)

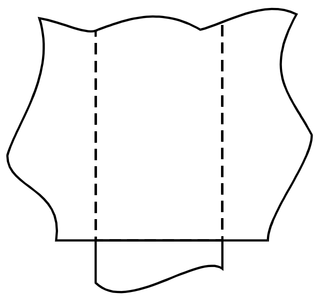


Fig. 31.1

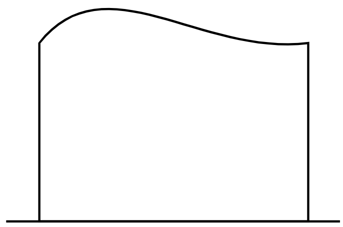


Fig. 31.2

Let us restrict our attention to finding the areas of such regions where the boundary is not familiar to us is on one side of x-axis only as in Fig. 31.2.

This is because we expect that it is possible to divide any region into a few subregions of this kind, find the areas of these subregions and finally add up all these areas to get the area of the whole region. (See Fig. 31.1)

Now, let f (x) be a continuous function defined on the closed interval [a, b]. For the present, assume that all the values taken by the function are non-negative, so that the graph of the function is a curve above the x-axis (See. Fig.31.3).

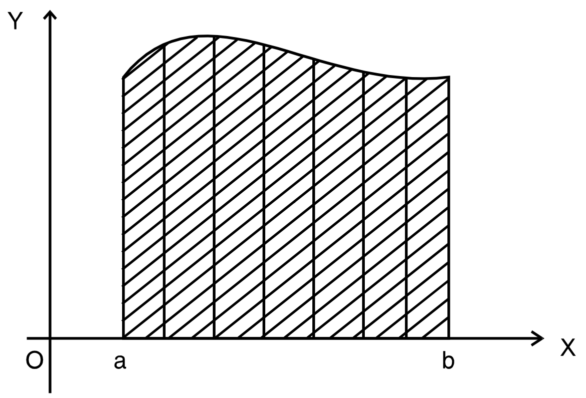


Fig. 31.3

Consider the region between this curve, the x-axis and the ordinates $x = a$ and $x = b$, that is, the shaded region in Fig.31.3. Now the problem is to find the area of the shaded region.

In order to solve this problem, we consider three special cases of $f(x)$ as rectangular region, triangular region and trapezoidal region.

The area of these regions $=$ base $\times$ average height

In general for any function $f(x)$ on $[a, b]$

Area of the bounded region (shaded region in Fig. 31.3) = base $\times$ average height

The base is the length of the domain interval [a, b]. The height at any point x is the value of $f(x)$ at that point. Therefore, the average height is the average of the values taken by f in [a, b]. (This may not be so easy to find because the height may not vary uniformly.) Our problem is how to find the average value of f in [a,b].

31.1.1 Average Value of a Function in an Interval

If there are only finite number of values of f in [a,b], we can easily get the average value by the formula.

$$\text{Average value of f in } [a, b] = \frac{\text{Sum of the values of f in } [a, b]}{\text{Numbers of values}}$$

But in our problem, there are infinite number of values taken by f in [a, b]. How to find the average in such a case? The above formula does not help us, so we resort to estimate the average value of f in the following way:

First Estimate : Take the value of f at 'a' only. The value of f at a is f (a). We take this value, namely f (a), as a rough estimate of the average value of f in [a,b].

Average value of f in [a, b] (first estimate) = f (a) (i)

Second Estimate : Divide [a, b] into two equal parts or sub-intervals.

Let the length of each sub-interval be h, $h = \dfrac{b-a}{2}$.

Take the values of f at the left end points of the sub-intervals. The values are $f(a)$ and $f(a+h)$ (Fig. 31.4)

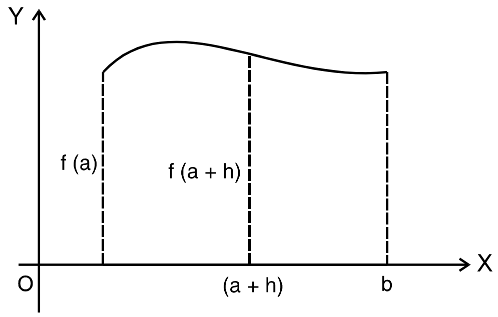


Fig. 31.4

Take the average of these two values as the average of f in [a, b].

Average value of f in [a, b] (Second estimate)

$$= \frac{f(a) + f(a+h)}{2}, \quad h = \frac{b-a}{2} \qquad \text{(ii)}$$

This estimate is expected to be a better estimate than the first.

Proceeding in a similar manner, divide the interval [a, b] into n subintervals of length h (Fig. 31.5), $h = \frac{b-a}{n}$

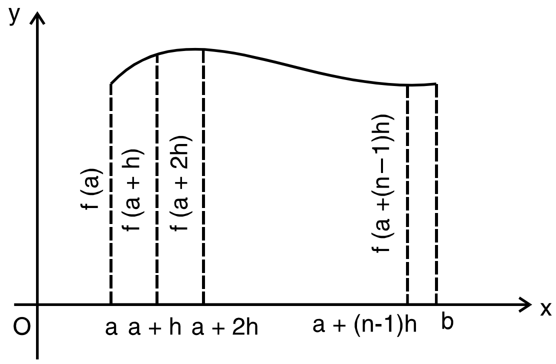


Fig. 31.5

Take the values of f at the left end points of the n subintervals.

The values are f (a), f (a + h),......,f [a + (n-1) h]. Take the average of these n values of f in [a, b].

Average value of f in [a, b] (nth estimate)

$$= \frac{f(a) + f(a+h) + \dots\dots\dots + f(a+(n-1)h)}{n}, \quad h = \frac{b-a}{n} \qquad \text{(iii)}$$

For larger values of n, (iii) is expected to be a better estimate of what we seek as the average value of f in [a, b]

Thus, we get the following sequence of estimates for the average value of f in [a, b]:

$$f(a)$$

$$\frac{1}{2}[f(a)+f(a+h)], \qquad h=\frac{b-a}{2}$$

$$\frac{1}{3}[f(a)+f(a+h)+f(a+2h)], \qquad h=\frac{b-a}{3}$$

....

....

$$\frac{1}{n}[f(a)+f(a+h)+........+f\{a+(n-1)h\}], \quad h=\frac{b-a}{n}$$

As we go farther and farther along this sequence, we are going closer and closer to our destination, namely, the average value taken by f in [a, b]. Therefore, it is reasonable to take the limit of these estimates as the average value taken by f in [a, b]. In other words,

Average value of f in [a, b]

$$\lim_{n\to\infty}\frac{1}{n}\{f(a)+f(a+h)+f(a+2h)+......+f[a+(n-1)h]\},$$

$$h=\frac{b-a}{n} \qquad \text{(iv)}$$

It can be proved that this limit exists for all continuous functions f on a closed interval [a, b].

Now, we have the formula to find the area of the shaded region in Fig. 31.3, The base is $(b-a)$ and the average height is given by (iv). The area of the region bounded by the curve f (x), x-axis, the ordinates x = a and x = b

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\{f(a)+f(a+h)+f(a+2h)+......+f[a+(n-1)h]\},$$

$$\lim_{n\to 0}\frac{1}{n}[f(a)+f(a+h)+........+f\{a+(n-1)h\}], h=\frac{b-a}{n} \qquad \text{(v)}$$

We take the expression on R.H.S. of (v) as the definition of a **definite integral.** This integral is denoted by

$$\int_a^b f(x)\,dx$$

read as integral of f (x) from a to b'. The numbers a and b in the symbol $\int_a^b f(x)\,dx$ are called respectively the lower and upper limits of integration, and f (x) is called the integrand.

Note : In obtaining the estimates of the average values of f in [a, b], we have taken the left end points of the subintervals. Why left end points?

Why not right end points of the subintervals? We can as well take the right end points of the subintervals throughout and in that case we get

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\{f(a+h)+f(a+2h)+......+f(b)\},\quad h=\frac{b-a}{n}$$

$$=\lim_{h\to 0} h\left[f(a+h)+f(a+2h)+......+f(b)\right] \qquad \text{(vi)}$$

Example 31.1 Find $\int_1^2 x\,dx$ as the limit of sum.

Solution : By definition,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+........+f\{a+(n-1)h\}\right],$$

$$h=\frac{b-a}{n}$$

Here a = 1, b = 2, f (x) = x and $h=\frac{1}{n}$.

$$\therefore \qquad \int_1^2 x\,dx = \lim_{n\to\infty}\frac{1}{n}\left[f(1)+f\left(1+\frac{1}{n}\right)+........+f\left(1+\frac{n-1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{1}{n}\right)+\left(1+\frac{2}{n}\right)........+\left(1+\frac{n-1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\underbrace{1+1+......+1}_{\text{n times}}+\left(\frac{1}{n}+\frac{2}{n}+........+\frac{n-1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}\left(1+2+......+(n-1)\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{(n-1).n}{n.2}\right]$$

$$\left[\text{Since } 1+2+3+....+(n-1)=\frac{(n-1).n}{2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{3n-1}{2}\right]$$

$$=\lim_{n\to\infty}\left[\frac{3}{2}-\frac{1}{2n}\right]=\frac{3}{2}$$

Example 31.2 Find $\int_0^2 e^x dx$ as limit of sum.

Solutions : By definition

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots.. + f\{a + (n-1)h\}\right]$$

where $\quad h = \dfrac{b-a}{n}$

Here $a = 0, b = 2, f(x) = e^x$ and $h = \dfrac{2-0}{n} = \dfrac{2}{n}$

$$\therefore \qquad \int_0^2 e^x dx = \lim_{h\to 0} h\left[f(0) + f(h) + f(2h) + \ldots\ldots. + f(n-1)h\right]$$

$$= \lim_{h\to 0} h\left[e^0 + e^h + e^{2h} + \ldots\ldots. + e^{(n-1)h}\right]$$

$$= \lim_{h\to 0} h\left[e^0\left(\frac{\left(e^h\right)^n - 1}{e^h - 1}\right)\right]$$

$$\left[\text{Since} \quad a + ar + ar^2 + \ldots\ldots. + ar^{n-1} = a\left(\frac{r^n - 1}{r-1}\right)\right]$$

$$= \lim_{h\to 0} h\left[\frac{e^{nh} - 1}{e^h - 1}\right] = \lim_{h\to 0} \frac{h}{h}\left[\frac{e^2 - 1}{\left(\dfrac{e^h - 1}{h}\right)}\right] \qquad (\because nh = 2)$$

$$= \lim_{h\to 0} \frac{e^2 - 1}{\dfrac{e^h - 1}{h}} = \frac{e^2 - 1}{1}$$

$$= e^2 - 1 \qquad \left[\because \lim_{h\to 0} \frac{e^h - 1}{h} = 1\right]$$

In examples 31.1 and 31.2 we observe that finding the definite integral as the limit of sum is quite difficult. In order to overcome this difficulty we have the fundamental theorem of integral calculus which states that

Theorem 1 : If f is continuous in [a, b] and F is an antiderivative of f in [a, b] then

$$\int_a^b \mathbf{f(x)\,dx = F(b) - F(a)} \qquad \ldots\ldots(1)$$

The difference F (b) – F (a) is commonly denoted by $\left[F(x)\right]_a^b$ so that (1) can be written as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \text{ or } \left[F(x)\right]_a^b$$

In words, the theorem tells us that

$$\int_a^b f(x)\,dx = \text{(Value of antiderivative at the upper limit b)}$$
$$\quad - \text{(Value of the same antiderivative at the lower limit a)}$$

Example 31.3 Evaluate the following

(a) $\int_0^{\frac{\pi}{2}} \cos x \, dx$ (b) $\int_0^2 e^{2x} \, dx$

Solution : We know that

$$\int \cos x \, dx = \sin x + c$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \cos x \, dx = [\sin x]_0^{\frac{\pi}{2}}$$

$$= \sin \frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

(b) $$\int_0^2 e^{2x} \, dx = \left[\frac{e^{2x}}{2} \right]_0^2, \qquad \left[\because \int e^x dx = e^x \right]$$

$$= \left(\frac{e^4 - 1}{2} \right)$$

Theorem 2 : If f and g are continuous functions defined in [a, b] and c is a constant then,

(i) $\int_a^b c f(x) dx = c \int_a^b f(x) dx$

(ii) $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$

(iii) $\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$

Example 31.4 Evaluate $\int_0^2 \left(4x^2 - 5x + 7 \right) dx$

Solution : $\int_0^2 \left(4x^2 - 5x + 7 \right) dx = \int_0^2 4x^2 dx - \int_0^2 5x \, dx + \int_0^2 7 \, dx$

$$= 4\int_0^2 x^2 dx - 5\int_0^2 x dx + 7\int_0^2 1\, dx$$

$$= 4.\left[\frac{x^3}{3}\right]_0^2 - 5\left[\frac{x^2}{2}\right]_0^2 + 7[x]_0^2$$

$$= 4.\left(\frac{8}{3}\right) - 5\left(\frac{4}{2}\right) + 7(2)$$

$$= \frac{32}{3} - 10 + 14$$

$$= \frac{44}{3}$$

CHECK YOUR PROGRESS 31.1

1. Find $\int_0^5 (x+1)\, dx$ as the limit of sum.

2. Find $\int_{-1}^1 e^x\, dx$ as the limit of sum.

3. Evaluate (a) $\int_0^{\frac{\pi}{4}} \sin x\, dx$ (b) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)\, dx$

   (c) $\int_0^1 \frac{1}{1+x^2}\, dx$ (d) $\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$

31.2 EVALUATION OF DEFINITE INTEGRAL BY SUBSTITUTION

The principal step in the evaluation of a definite integral is to find the related indefinite integral. In the preceding lesson we have discussed several methods for finding the indefinite integral. One of the important methods for finding indefinite integrals is the method of substitution. When we use substitution method for evaluation the definite integrals, like

$$\int_2^3 \frac{x}{1+x^2}\, dx, \quad \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\, dx,$$

the steps could be as follows :

(i) Make appropriate substitution to reduce the given integral to a known form to integrate. Write the integral in terms of the new variable.

(ii) Integrate the new integrand with respect to the new variable.

(iii) Change the limits accordingly and find the difference of the values at the upper and lower limits.

Note : If we don't change the limit with respect to the new variable then after integrating resubstitute for the new variable and write the answer in original variable. Find the values of the answer thus obtained at the given limits of the integral.

Example 31.5 Evaluate the following :

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}\,d\theta$ (c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}$

Solution : (a) Let $\cos x = t$ then $\sin x\,dx = -dt$

When $x = 0$, $t = 1$ and $x = \frac{\pi}{2}, t = 0$. As x varies from 0 to $\frac{\pi}{2}$, t varies from 1 to 0.

$$\therefore \qquad \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_1^0 \frac{1}{1+t^2}\,dt = -\left[\tan^{-1} t\right]_1^0$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= -\left[0 - \frac{\pi}{4}\right] = \frac{\pi}{4}$$

(b) $$I = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}\,d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\left(\sin^2\theta+\cos^2\theta\right)^2 - 2\sin^2\theta\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1-2\sin^2\theta\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta\,d\theta}{1-2\sin^2\theta\left(1-\sin^2\theta\right)}$$

Let $\sin^2\theta = t$

Then $2\sin\theta\cos\theta\,d\theta = dt$ i.e. $\sin 2\theta\,d\theta = dt$

When $\theta = 0, t = 0$ and $\theta = \frac{\pi}{2}, t = 1$. As θ varies from 0 to $\frac{\pi}{2}$, the new variable t varies from 0 to 1.

$$\therefore \qquad I = \int_0^1 \frac{1}{1-2t(1-t)}\,dt = \int_0^1 \frac{1}{2t^2-2t+1}\,dt$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{t^2 - t + \frac{1}{4} + \frac{1}{4}}\,dt \qquad I = \frac{1}{2}\int_0^1 \frac{1}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\,dt$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{1}{2}}\left[\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1 = \left[\tan^{-1}1 - \tan^{-1}(-1)\right]$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

(c) We know that $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\therefore \qquad \int_0^{\frac{\pi}{2}} \frac{1}{5+4\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{5 + \dfrac{4\left(1-\tan^2\left(\frac{x}{2}\right)\right)}{\left(1+\tan^2\left(\frac{x}{2}\right)\right)}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{9+\tan^2\left(\frac{x}{2}\right)}\,dx \qquad (1)$$

Let $\qquad \tan\dfrac{x}{2} = t$

Then $\quad \sec^2\dfrac{x}{2}\,dx = 2dt$ when $x = 0,\ t = 0$, when $x = \dfrac{\pi}{2},\ t = 1$

$$\therefore \qquad \int_0^{\frac{\pi}{2}} \frac{1}{5+4\cos x}\,dx = 2\int_0^1 \frac{1}{9+t^2}\,dt \qquad \text{[From (1)]}$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^1 = \frac{2}{3}\left[\tan^{-1}\frac{1}{3}\right]$$

31.3 SOME PROPERTIES OF DEFINITE INTEGRALS

The definite integral of f (x) between the limits a and b has already been defined as

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ Where } \frac{d}{dx}[F(x)] = f(x),$$

where a and b are the lower and upper limits of integration respectively. Now we state below some important and useful properties of such definite integrals.

(i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ (ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$ where $a<c<b$.

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(v) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(vi) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(vii) $$\int_0^{2a} f(x)\,dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \end{cases}$$

(viii) $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is an odd function of } x \\ 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function of } x \end{cases}$$

Many of the definite integrals may be evaluated easily with the help of the above stated properties, which could have been very difficult otherwise.

The use of these properties in evaluating definite integrals will be illustrated in the following examples.

Example 31.6 Show that

(a) $\int_0^{\frac{\pi}{2}} \log|\tan x|\,dx = 0$ (b) $\int_0^{\pi} \frac{x}{1+\sin x}\,dx = \pi$

Solution : (a) Let $\quad I = \int_0^{\frac{\pi}{2}} \log |\tan x|\, dx$ ....(i)

Using the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$$I = \int_0^{\frac{\pi}{2}} \log\left(\tan\left(\frac{\pi}{2} - x\right)\right) dx = \int_0^{\frac{\pi}{2}} \log(\cot x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\tan x)^{-1}\,dx = -\int_0^{\frac{\pi}{2}} \log \tan x\,dx$$

$$= -I \qquad \text{[Using (i)]}$$

$\therefore \qquad 2I = 0$

i.e. $\qquad I = 0 \quad$ or $\quad \int_0^{\frac{\pi}{2}} \log|\tan x|\,dx = 0$

(b) $\qquad \int_0^{\pi} \frac{x}{1+\sin x}\,dx$

Let $\qquad I = \int_0^{\pi} \frac{x}{1+\sin x}\,dx \qquad$ (i)

$\therefore \qquad I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx \qquad \text{(ii)}$$

Adding (i) and (ii)

$$2I = \int_0^{\pi} \frac{x + \pi - x}{1+\sin x}\,dx = \pi\int_0^{\pi} \frac{1}{1+\sin x}\,dx$$

or $\qquad 2I = \pi\int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}\,dx$

$$= \pi\int_0^{\pi} \left(\sec^2 x - \tan x \sec x\right) dx$$

$$= \pi\left[\tan x - \sec x\right]_0^{\pi}$$

$$= \pi\left[\left(\tan \pi - \sec \pi\right) - \left(\tan 0 - \sec 0\right)\right]$$

$$= \pi\left[0 - (-1) - (0 - 1)\right]$$

$$= 2\pi$$

$\therefore \quad I = \pi$

Example 31.7 Evaluate

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx$

Solution : (a) Let $\displaystyle I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ (i)

Also $$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\, dx$$

(Using the property $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$).

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \int_0^{\frac{\pi}{2}} 1.dx$$

$$= \left[x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{\pi}{4}$$

i.e. $$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi}{4}$$

(b) Let $$I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x}\,dx \qquad \text{(i)}$$

Then $$I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\cos x \sin x}\,dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x} + \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x + \cos x - \sin x}{1+\sin x \cos x}\,dx$$

$$= 0$$

$\therefore$ $$I = 0$$

Example 31.8 Evaluate (a) $\displaystyle\int_{-a}^{a} \frac{xe^{x^2}}{1+x^2}\,dx$ (b) $\displaystyle\int_{-3}^{3} |x+1|\,dx$

Solution : (a) Here $$f(x) = \frac{xe^{x^2}}{1+x^2} \qquad \therefore \qquad f(-x) = -\frac{xe^{x^2}}{1+x^2}$$

$$= -f(x)$$

$\therefore$ $f(x)$ is an odd function of x.

$\therefore$ $$\int_{-a}^{a} \frac{xe^{x^2}}{1+x^2}\,dx = 0$$

(b) $\int_{-3}^{3} |x+1|\,dx$

$$|x+1| = \begin{cases} x+1, \text{ if } x \ge -1 \\ -x-1, \text{ if } x < -1 \end{cases}$$

$$\therefore \int_{-3}^{3} |x+1|\,dx = \int_{-3}^{-1} |x+1|\,dx + \int_{-1}^{3} |x+1|\,dx, \text{ using property (iii)}$$

$$= \int_{-3}^{-1} (-x-1)\,dx + \int_{-1}^{3} (x+1)\,dx$$

$$= \left[\frac{-x^2}{2} - x\right]_{-3}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{3}$$

$$= -\frac{1}{2} + 1 + \frac{9}{2} - 3 + \frac{9}{2} + 3 - \frac{1}{2} + 1 = 10$$

Example 31.9 Evaluate $\int_{0}^{\frac{\pi}{2}} \log(\sin x)\,dx$

Solution : Let $I = \int_{0}^{\frac{\pi}{2}} \log(\sin x)\,dx$ (i)

Also $I = \int_{0}^{\frac{\pi}{2}} \log\left[\sin\left(\frac{\pi}{2} - x\right)\right]dx,$ [Using property (iv)]

$$= \int_{0}^{\frac{\pi}{2}} \log(\cos x)\,dx \quad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_{0}^{\frac{\pi}{2}} [\log(\sin x) + \log(\cos x)]\,dx = \int_{0}^{\frac{\pi}{2}} \log(\sin x \cos x)\,dx$$

$$= \int_{0}^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right)dx = \int_{0}^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \int_{0}^{\frac{\pi}{2}} \log(2)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \frac{\pi}{2}\log 2 \qquad \text{(iii)}$$

Again, let $\quad I_1 = \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx$

Put $2x = t \quad \Rightarrow \quad dx = \frac{1}{2}dt$

When $x = 0$, $t = 0$ and $x = \frac{\pi}{2}, t = \pi$

$$\therefore \qquad I_1 = \frac{1}{2}\int_0^{\pi} \log(\sin t)\,dt$$

$$= \frac{1}{2}.2\int_0^{\frac{\pi}{2}} \log(\sin t)\,dt, \qquad \text{[using property (vi)]}$$

$$= \frac{1}{2}.2\int_0^{\frac{\pi}{2}} \log(\sin x)\,dt \qquad \text{[using property (i)]}$$

$\therefore \qquad I_1 = I, \qquad$ [from (i)] $\qquad$ .....(iv)

Putting this value in (iii), we get

$$2I = I - \frac{\pi}{2}\log 2 \quad \Rightarrow \quad I = -\frac{\pi}{2}\log 2$$

Hence, $\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx = -\frac{\pi}{2}\log 2$

CHECK YOUR PROGRESS 31.2

Evaluate the following integrals :

1. $\int_0^1 xe^{x^2}dx$
2. $\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\sin x}$
3. $\int_0^1 \frac{2x+3}{5x^2+1}dx$
4. $\int_{-5}^{5} |x+2|\,dx$
5. $\int_0^2 x\sqrt{2-x}\,dx$
6. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x}dx$

7. $\int_0^{\frac{\pi}{2}} \log \cos x \, dx$ 8. $\int_{-a}^{a} \frac{x^3 e^{x^4}}{1+x^2} dx$ 9. $\int_0^{\frac{\pi}{2}} \sin 2x \log \tan x \, dx$

10. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x+\cos x} dx$

31.4 APPLICATIONS OF INTEGRATION

Suppose that f and g are two continuous functions on an interval [a, b] such that $f(x) \leq g(x)$ for $x \in [a, b]$ that is, the curve y = f (x) does not cross under the curve y = g (x) over [a, b]. Now the question is how to find the area of the region bounded above by y = f (x), below by y = g (x), and on the sides by x = a and x = b.

Again what happens when the upper curve y = f (x) intersects the lower curve y = g (x) at either the left hand boundary x = a , the right hand boundary x = b or both?

31.4.1 Area Bounded by the Curve, x-axis and the Ordinates

Let AB be the curve y = f (x) and CA, DB the two ordinates at x = a and x = b respectively. Suppose y = f (x) is an increasing function of x in the interval $a \leq x \leq b$.

Let P (x, y) be any point on the curve and $Q(x+\delta x, y+\delta y)$ a neighbouring point on it. Draw their ordinates PM and QN.

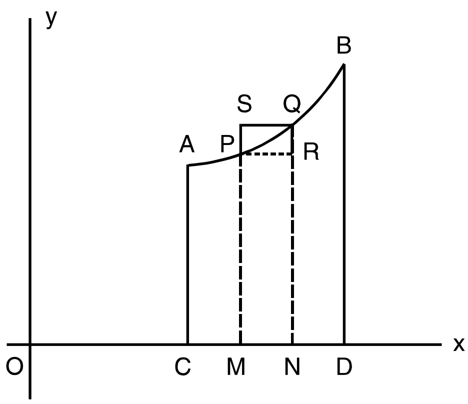


Fig. 31.6

Here we observe that as x changes the area (ACMP) also changes. Let

$$A = \text{Area (ACMP)}$$

Then the area (ACNQ) $= A + \delta A$.

The area (PMNQ)=Area (ACNQ) – Area (ACMP)

$$= A + \delta A - A = \delta A.$$

Complete the rectangle PRQS. Then the area (PMNQ) lies between the areas of rectangles PMNR and SMNQ, that is

δA lies between $y \ \delta x$ and $(y+\delta y)\delta x$

$\Rightarrow$ $\frac{\delta A}{\delta x}$ lies between y and $(y+\delta y)$

In the limiting case when $Q \to P$, $\delta x \to 0$ and $\delta y \to 0$.

$$\therefore \qquad \lim_{\delta x \to 0} \frac{\delta A}{\delta x} \text{ lies between } y \text{ and } \lim_{\delta y \to 0} (y + \delta y)$$

$$\therefore \qquad \frac{dA}{dx} = y$$

Integrating both sides with respect to x, from x = a to x = b, we have

$$\int_a^b y\,dx = \int_a^b \frac{dA}{dx} \cdot dx = [A]_a^b$$

= (Area when x = b) – (Area when x = a)

= Area (ACDB) – 0

= Area (ACDB).

Hence $\quad$ Area (ACDB) $= \int_a^b f(x)\,dx$

The area bounded by the curve y = f (x), the x-axis and the ordinates x = a, x = b is

$$\int_a^b f(x)\,dx \text{ or } \int_a^b y\,dx$$

where y = f (x) is a continuous single valued function and y does not change sign in the interval $a \le x \le b$.

Example31.10 Find the area bounded by the curve y = x, x-axis and the lines x =0, x = 2.

Solution : The given curve is y = x

$\therefore$ Required area bounded by the curve, x-axis and the ordinates $x = 0,\ x = 2$ (as shown in Fig.31.7)

is $$\int_0^2 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$$= 2 - 0 = 2 \text{ square units}$$

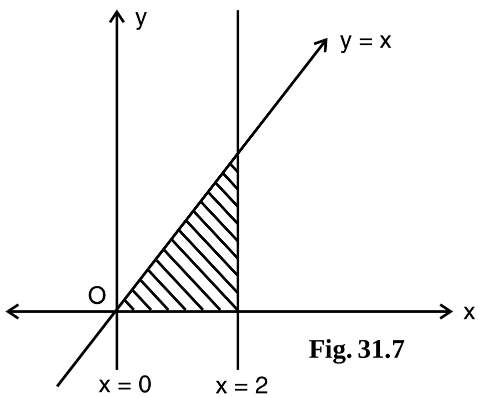


Fig. 31.7

Example 31.11 Find the area enclosed by the circle $x^2 + y^2 = a^2$, and x-axis in the first quadrant.

Solution : The given curve is $x^2 + y^2 = a^2$, which is a circle whose centre and radius are (0, 0) and a respectively. Therefore, we have to find the area enclosed by the circle $x^2 + y^2 = a^2$, the x-axis and the ordinates x = 0 and x = a.

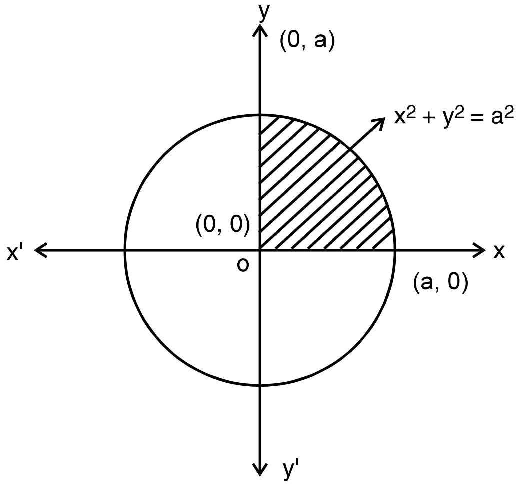


Fig. 31.8

$\therefore$ Required area $= \int_0^a y\,dx$

$$= \int_0^a \sqrt{a^2 - x^2}\,dx,$$

($\because$ y is positive in the first quadrant)

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1}1 - 0 - \frac{a^2}{2}\sin^{-1}0$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} \quad \left(\because \sin^{-1}1 = \frac{\pi}{2}, \sin^{-1}0 = 0\right)$$

$$= \frac{\pi a^2}{4} \text{ square units}$$

CHECK YOUR PROGRESS 31.3

1. Find the area bounded by the curve $y = x^2$, x-axis and the lines $x = 0, x = 2$.
2. Find the area bounded by the curve y = 3 x, x-axis and the lines x = 0 and x = 3.

31.4.2. Area Bounded by the Curve x = f (y) between y-axis and the Lines y = c, y = d

Let AB be the curve x = f (y) and let CA, DB be the abscissae at y = c, y = d respectively.

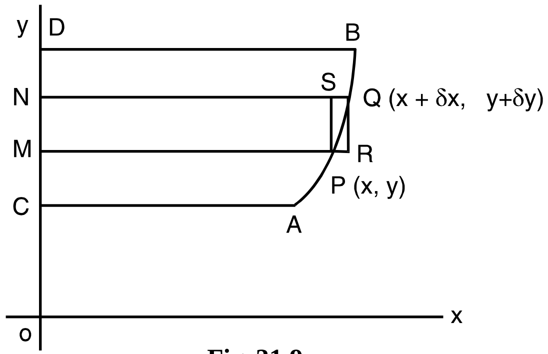


Fig. 31.9

Let P (x, y) be any point on the curve and let $Q(x + \delta x, y + \delta y)$ be a neighbouring point on it. Draw PM and QN perpendiculars on y-axis from P and Q respectively. As y changes, the area (ACMP) also changes and hence clearly a function of y. Let A denote the area (ACMP), then the area (ACNQ) will be $A + \delta A$.

The area (PMNQ) = Area (ACNQ) $-$ Area (ACMP) $= A + \delta A - A = \delta A$.

Complete the rectangle PRQS. Then the area (PMNQ) lies between the area (PMNS) and the area (RMNQ), that is,

δA lies between $x\ \delta\ y$ and $(x+\delta x)\,\delta y$

$$\Rightarrow \qquad \frac{\delta A}{\delta y} \text{ lies between } x \text{ and } x+\delta x$$

In the limiting position when $Q \to P,\ \delta x \to 0$ and $\therefore$.

$$\therefore \qquad \lim_{\delta y \to 0} \frac{\delta A}{\delta y} \text{ lies between } x \text{ and } \lim_{\delta x \to 0}(x+\delta x)$$

$$\Rightarrow \qquad \Rightarrow \frac{dA}{dy} = x$$

Integrating both sides with respect to y, between the limits c to d, we get

$$\int_c^d x\,dy = \int_c^d \frac{dA}{dy}\cdot dy$$

$$= = [A]_c^d$$

= (Area when y = d) – (Area when y = c)
= Area (ACDB) – 0
= Area (ACDB)

Hence area (ACDB) $= \int_c^d x\,dy = \int_c^d f(y)\,dy$

The area bounded by the curve $x = f(y)$, the y-axis and the lines $y = c$ and $y = d$ is

$$\int_c^d x\,dy \quad \text{or} \quad \int_c^d f(y)\,dy$$

where $x = f(y)$ is a continuous single valued function and x does not change sign in the interval $c \le y \le d$.

Example 31.12 Find the area bounded by the curve x = y, y-axis and the lines y = 0, y = 3.

Solution : The given curve is x = y.

$\therefore$ Required area bounded by the curve, y-axis and the lines y =0, y = 3 is

$$= \int_0^3 x\,dy$$

$$= \int_0^3 y\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^3$$

$$= \frac{9}{2} - 0$$

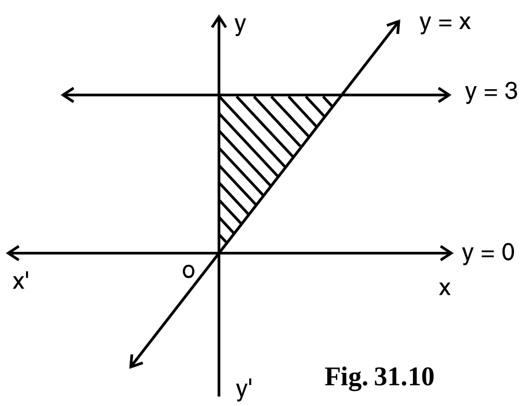


Fig. 31.10

$$= \frac{9}{2} \text{ square units}$$

Example 31.13 Find the area enclosed by the circle $x^2 + y^2 = a^2$ and y-axis in the first quadrant.

Solution : The given curve is $x^2 + y^2 = a^2$, which is a circle whose centre is (0, 0) and radius a. Therefore, we have to find the area enclosed by the circle $x^2 + y^2 = a^2$, the y-axis and the abscissae y = 0, y = a.

$$\therefore \quad \text{Required area} = \int_0^a x\, dy$$

$$= \int_0^a \sqrt{a^2 - y^2}\, dy$$

(because x is positive in first quadrant)

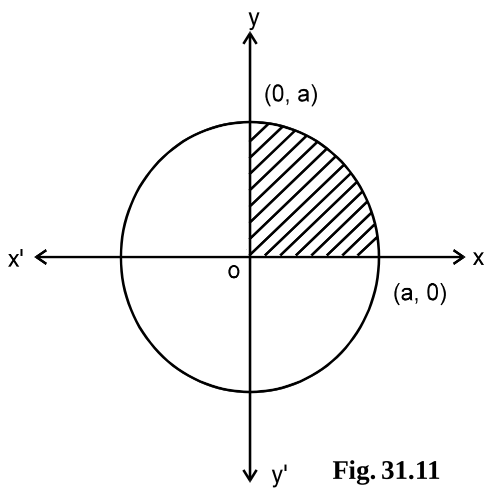


Fig. 31.11

$$= \left[\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)\right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1}1 - 0 - \frac{a^2}{2}\sin^{-1}0$$

$$= \frac{\pi a^2}{4} \text{ square units} \qquad \left(\because \sin^{-1}0 = 0, \sin^{-1}1 = \frac{\pi}{2}\right)$$

Note : The area is same as in Example 31.11, the reason is the given curve is symmetrical about both the axes. In such problems if we have been asked to find the area of the curve, without any restriction we can do by either method.

Example 31.14 Find the whole area bounded by the circle $x^2 + y^2 = a^2$.

Solution : The equation of the curve is $x^2 + y^2 = a^2$.

The circle is symmetrical about both the axes, so the whole area of the circle is four times the area os the circle in the first quadrant, that is,

Area of circle = 4 × area of OAB

$$= 4 \times \frac{\pi a^2}{4} \text{(From Example 12.11 and 12.13)} = \pi a^2$$

square units

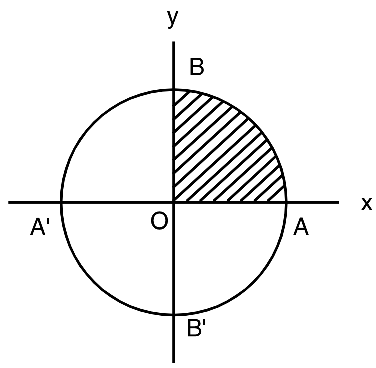
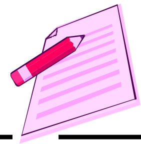


Fig. 31.12



Example 31.15 Find the whole area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Solution : The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The ellipse is symmetrical about both the axes and so the whole area of the ellipse is four times the area in the first quadrant, that is,
Whole area of the ellipse $= 4\times$ area (OAB)

In the first quadrant,

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2} \quad \text{or} \quad y=\frac{b}{a}\sqrt{a^2-x^2}$$

Now for the area (OAB), x varies from 0 to a

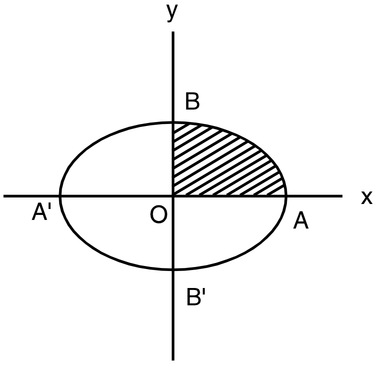


Fig. 31.13

$$\therefore \qquad \text{Area (OAB)} = \int_0^a y\,dx$$

$$=\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$=\frac{b}{a}\left[0+\frac{a^2}{2}\sin^{-1}1-0-\frac{a^2}{2}\sin^{-1}0\right]$$

$$=\frac{ab\pi}{4}$$

Hence the whole area of the ellipse

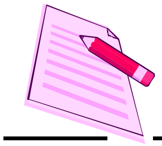
$$=4\times\frac{ab\pi}{4}$$

$$=\pi ab. \text{ square units}$$

31.4.3 Area between two Curves

Suppose that f (x) and g (x) are two continuous and non-negative functions on an interval [a, b] such that $f(x)\geq g(x)$ for all $x\in[a,\ b]$ that is, the curve y = f (x) does not cross under the curve y = g (x) for $x\in[a, b]$. We want to find the area bounded above by y = f (x), below by y = g (x), and on the sides by x = a and x = b.

MODULE - VIII
Calculus



Notes

Let A= [Area under y = f (x)] – [Area under y = g (x)] .....(1)

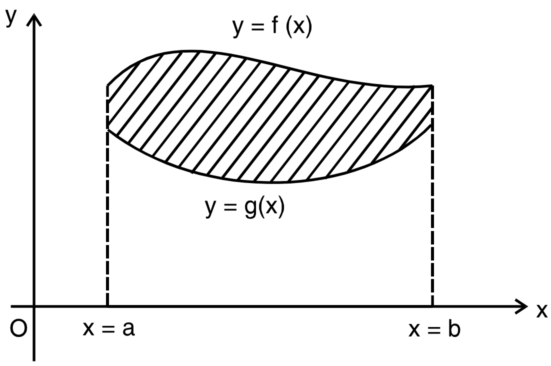


Fig. 31.14

Now using the definition for the area bounded by the curve $y = f(x)$, x-axis and the ordinates x = a and x = b, we have

Area under

$$y = f(x) = \int_a^b f(x)\,dx \qquad .....(2)$$

Similarly, Area under $y = g(x) = \int_a^b g(x)\,dx$ .....(3)

Using equations (2) and (3) in (1), we get

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx \qquad .....(4)$$

What happens when the function g has negative values also? This formula can be extended by translating the curves f (x) and g (x) upwards until both are above the x-axis. To do this let-m be the minimum value of g (x) on [a, b] (see Fig. 31.15).

Since $\quad g(x) \geq -m \quad \Rightarrow \quad g(x) + m \geq 0$

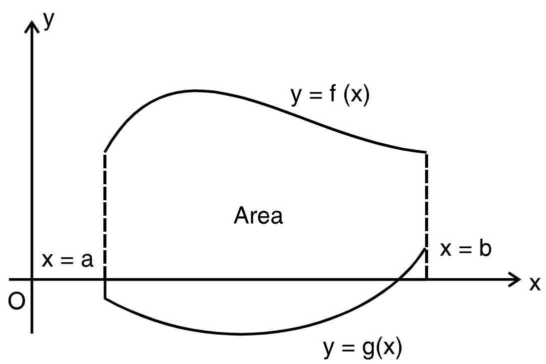


Fig. 31.15

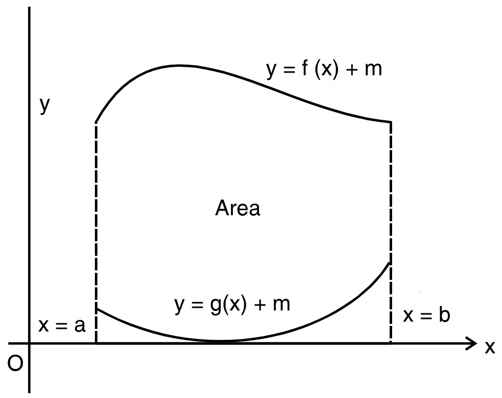


Fig. 31.16

Now, the functions $g(x) + m$ and $f(x) + m$ are non-negative on [a, b] (see Fig. 31.16). It is intuitively clear that the area of a region is unchanged by translation, so the area A between f and g is the same as the area between $g(x) + m$ and $f(x) + m$. Thus, A = [area under $y = [f(x)+m]$] – [area under $y = [g(x)+m]$] .....(5)

Now using the definitions for the area bounded by the curve y = f (x), x-axis and the ordinates x = a and x = b, we have

$$\text{Area under } y = f(x)+m = \int_a^b [f(x)+m]\,dx \qquad .....(6)$$

and $$\text{Area under } y = g(x)+m = \int_a^b [g(x)+m]\,dx \qquad (7)$$

The equations (6), (7) and (5) give

$$A = \int_a^b [f(x)+m]\,dx - \int_a^b [g(x)+m]\,dx$$

$$= \int_a^b [f(x)-g(x)]\,dx$$

which is same as (4) Thus,

If f (x) and g (x) are continuous functions on the interval [a, b], and $f(x) \geq g(x), \forall x \in [a, b]$, then the area of the region bounded above by y = f (x), below by y = g (x), on the left by x = a and on the right by x = b is

$$= \int_a^b [f(x)-g(x)]\,dx$$

$$= \frac{34}{3} \text{ square units}$$

If the curves intersect then the sides of the region where the upper and lower curves intersect reduces to a point, rather than a vertical line segment.

Example 31.16 Find the area of the region enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution : We know that $y = x^2$ is the equation of the parabola which is symmetric about the y-axis and vertex is origin and y = x + 6 is the equation of the straight line. (See Fig. 31.17).

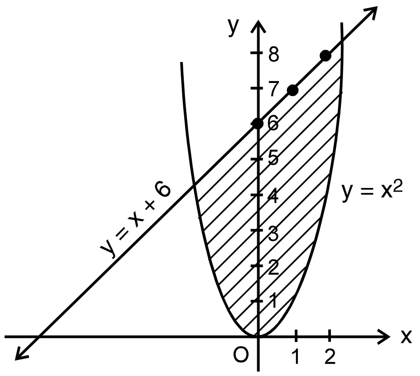


Fig. 31.17

A sketch of the region shows that the lower boundary is $y = x^2$ and the upper boundary is y = x +6. These two curves intersect at two points, say A and B. Solving these two equations we get

$$x^2 = x + 6 \quad \Rightarrow \quad x^2 - x - 6 = 0$$

$$\Rightarrow \quad (x-3)(x+2) = 0 \quad \Rightarrow \quad x = 3, -2$$

When x = 3, y = 9 and when x = –2, y = 4

$$\therefore \text{ The required area } = \int_{-2}^{3} \left[(x+6) - x^2 \right] dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_{-2}^{3}$$

$$= \frac{27}{2} - \left(-\frac{22}{3} \right)$$

$$= \frac{125}{6} \text{ square units}$$

Example 31.17 Find the area bounded by the curves $y^2 = 4x$ and y = x.

Solution : We know that $y^2 = 4x$ the equation of the parabola which is symmetric about the x-axis and origin is the vertex. y = x is the equation of the straight line (see Fig. 31.18).

A sketch of the region shows that the lower boundary is y = x and the upper boundary is $y^2 = 4x$. These two curves intersect at two points O and A. Solving these two equations, we get

$$\frac{y^2}{4} - y = 0$$

$$\Rightarrow \quad y(y-4) = 0$$

$$\Rightarrow \quad y = 0, 4$$

When y = 0, x = 0 and when y = 4, x = 4.

Here $f(x) = (4x)^{\frac{1}{2}}, g(x) = x, a = 0, b = 4$

Therefore, the required area is

$$= \int_0^4 \left(2x^{\frac{1}{2}} - x \right) dx$$

$$= \left[\frac{4}{3} x^{\frac{3}{2}} - \frac{x^2}{2} \right]_0^4$$

$$= \frac{32}{3} - 8$$

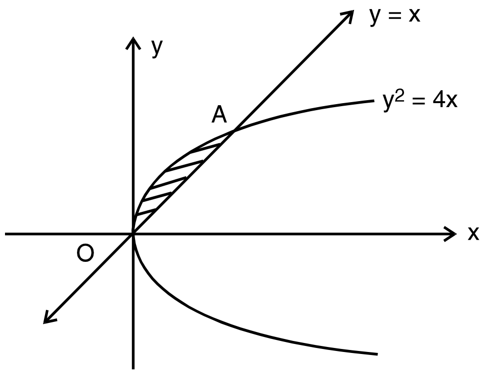


Fig. 31.18

$$= \frac{8}{3} \quad \text{square units}$$

Example 31.18 Find the area common to two parabolas $x^2 = 4ay$ and $y^2 = 4ax$.

Solution : We know that $y^2 = 4ax$ and $x^2 = 4ay$ are the equations of the parabolas, which are symmetric about the x-axis and y-axis respectively.

Also both the parabolas have their vertices at the origin (see Fig. 31.19).

A sketch of the region shows that the lower boundary is $x^2 = 4ay$ and the upper boundary is $y^2 = 4ax$. These two curves intersect at two points O and A. Solving these two equations, we have

$$\frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow \quad x\left(x^3 - 64a^3\right) = 0$$

$$\Rightarrow \quad x = 0, 4a$$

Hence the two parabolas intersect at point (0, 0) and (4a, 4a).

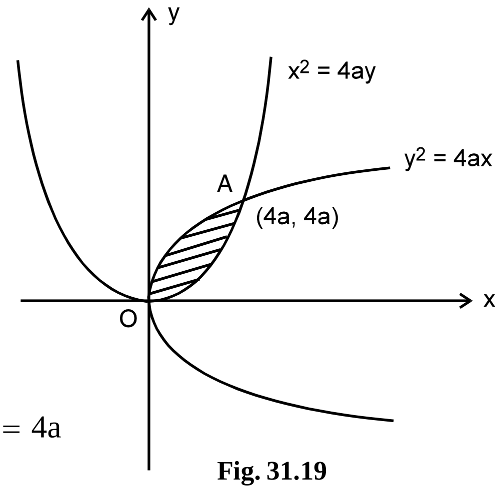


Fig. 31.19

Here $\quad f(x) = \sqrt{4ax}, g(x) = \frac{x^2}{4a}, a = 0 \text{ and } b = 4a$

Therefore, required area

$$= \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right] dx$$

$$= \left[\frac{2.2\sqrt{a}\,x^{\frac{3}{2}}}{3} - \frac{x^3}{12a}\right]_0^{4a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16}{3} a^2 \text{ square units}$$

CHECK YOUR PROGRESS 31.4

1. Find the area of the circle $x^2 + y^2 = 9$

2. Find the area of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

3. Find the area of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

4. Find the area bounded by the curves $y^2 = 4ax$ and $y = \frac{x^2}{4a}$

5. Find the area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$.

6. Find the area enclosed by the curves $y = x^2$ and $y = x + 2$

LET US SUM UP

- If f is continuous in [a, b] and F is an anti derivative of f in [a, b], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- If f and g are continuous in [a, b] and c is a constant, then

(i) $$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

(ii) $$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(iii) $$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

- The area bounded by the curve y = f (x), the x-axis and the ordinates

$$x = a,\ x = b \text{ is } \int_a^b f(x)\,dx \text{ or } \int_a^b y\,dx$$

where $y = f(x)$ is a continuous single valued function and y does not change sign in the interval $a \le x \le b$

- If f (x) and g (x) are continuous functions on the interval [a, b] and $f(x) \geq g(x)$, for all $x \in [a, b]$, then the area of the region bounded above by y = f (x), below by y = g (x), on the left by x = a and on the right by x = b is

$$\int_a^b [f(x) - g(x)]\,dx$$

SUPPORTIVE WEB SITES

http://mathworld.wolfram.com/DefiniteIntegral.html

http://www.mathsisfun.com/calculus/integration-definite.html

TERMINAL EXERCISE

Evaluate the following integrals (1 to 5) as the limit of sum.

1. $\int_a^b x\,dx$
2. $\int_a^b x^2\,dx$
3. $\int_0^2 \left(x^2 + 1\right)dx$

Evaluate the following integrals (4 to 20)

4. $\int_0^2 \sqrt{a^2 - x^2}\,dx$
5. $\int_0^{\frac{\pi}{2}} \sin 2x\,dx$
6. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\,dx$
7. $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$
8. $\int_0^1 \sin^{-1} x\,dx$
9. $\int_0^1 \frac{1}{\sqrt{1 - x^2}}\,dx$
10. $\int_3^4 \frac{1}{x^2 - 4}\,dx$
11. $\int_0^{\pi} \frac{1}{5 + 3\cos\theta}\,d\theta$
12. $\int_0^{\frac{\pi}{4}} 2\tan^3 x\,dx$
13. $\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$
14. $\int_0^2 x\sqrt{x + 2}\,dx$
15. $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\cos^5\theta\,d\theta$
16. $\int_0^{\pi} x \log \sin x\,dx$
17. $\int_0^{\pi} \log(1 + \cos x)\,dx$
18. $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx$

19. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$

20. $\int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx$

21. Find the area bounded by the curve $x = y^2$, $y-$axis and lines y=0, y=2.

22. Find the area of the region bounded by the curve $y = x^2$ and y=x.

23. Find the area bounded by the curve $y^2 = 4x$ and straight line x=3.

24. Find the area of triangular region whose vertices have coordinates (1,0), (2,2) and (3.1)

25. Find the area of the smaller region bounded by the cllipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the straight line $\frac{x}{3} + \frac{y}{2} = 1$

26. Find the area of the region bounded by the paralal $y = x^2$ and the curve $y = |x|$

ANSWERS

CHECK YOUR PROGRESS 31.1

1. $\frac{35}{2}$ 2. $e - \frac{1}{e}$

3. (a) $\frac{\sqrt{2}-1}{\sqrt{2}}$ (b) 2 (c) $\frac{\pi}{4}$ (d) $\frac{64}{3}$

CHECK YOUR PROGRESS 31.2

1. $\frac{e-1}{2}$ 2. $\frac{2}{3}\tan^{-1}\frac{1}{3}$ 3. $\frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$

4. 29 5. $\frac{24\sqrt{2}}{15}$ 6. $\frac{\pi}{4}$

7. $-\frac{\pi}{2}\log 2$ 8. 0 9. 0 10. $\frac{1}{2}\left[\frac{\pi}{2} - \log 2\right]$

CHECK YOUR PROGRESS 31.3

1. $\frac{8}{3}$ sq. units 2. $\frac{27}{2}$ sq. units

CHECK YOUR PROGRESS 31.4

1. 9π sq. units 2. 6π sq. units 3. 20π sq. units

4. $\frac{16}{3}a^2$ sq. units 5. $\frac{16}{3}$ sq. units 6. $\frac{9}{2}$ sq. units

TERMINAL EXERCISE

1. $\frac{b^2 - a^2}{2}$ 2. $\frac{b^3 - a^3}{3}$ 3. $\frac{14}{3}$ 4. $\frac{\pi a^2}{4}$

5. 1 6. $\frac{1}{2}\log 2$ 7. $\frac{\pi}{4}$ 8. $\frac{\pi}{2} - 1$

9. $\frac{\pi}{2}$ 10. $\frac{1}{4}\log\frac{5}{3}$ 11. $\frac{\pi}{4}$ 12. $1 - \log 2$

13. $\frac{2}{3}$

14. $\frac{16}{15}\left(2+\sqrt{2}\right)$

15. $\frac{64}{231}$

16. $-\frac{\pi^2}{2}\log 2$

17. $-\pi \log 2$

18. $\frac{\pi^2}{4}$

19. $\frac{1}{\sqrt{2}}\log\left(1+\sqrt{2}\right)$

20. $\frac{\pi}{8}\log 2$

21. $\frac{8}{3}$ square unit.

22. $\frac{1}{6}$ Square unit

23. $8\sqrt{3}$ Square unit

24. $\frac{3}{2}$ Square unit

25. $\frac{3}{2}(\pi - 2)$ Square unit

26. $\frac{1}{3}$ Square unit