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APPLICATIONS OF DERIVATIVES

In the previous lesson, we have learnt that the slope of a line is the tangent of the angle which the line makes with the positive direction of x-axis. It is denoted by the letter 'm'. Thus, if θ is the angle which a line makes with the positive direction of x-axis, then m is given by tan θ.

We have also learnt that the slope m of a line, passing through two points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$

In this lesson, we shall find the equations of tangents and normals to different curves, using derinatives.

OBJECTIVES

After studying this lesson, you will be able to :

- find rate of change of quantities
- find approximate value of functions
- define tangent and normal to a curve (graph of a function) at a point;
- find equations of tangents and normals to a curve under given conditions;
- define monotonic (increasing and decreasing) functions;
- establish that $\frac{dy}{dx} > 0$ in an interval for an increasing function and $\frac{dy}{dx} < 0$ for a decreasing function;
- define the points of maximum and minimum values as well as local maxima and local minima of a function from the graph;
- establish the working rule for finding the maxima and minima of a function using the first and the second derivatives of the function; and
- work out simple problems on maxima and minima.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of coordinate geometry and
- Concept of tangent and normal to a curve
- Concept of diferential coefficient of various functions
- Geometrical meaning of derivative of a function at a point
- Solution of equetions and the inequations.

29.1 RATE OF CHANGE OF QUANTITIES

Let $y = f(x)$ be a function of x and let there be a small change Δx in x, and the corresponding change in y be Δy.

$\therefore$ Average change in y per unit change in $x = \dfrac{\Delta y}{\Delta x}$

As $\Delta x \to 0$, the limiting value of the average rate of change of y with respect to x.

So the rate of change of y per unit change in x

$$= \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Hence, $\dfrac{dy}{dx}$ represents the rate of change of y with respect to x.

Thus,

The value of $\dfrac{dy}{dx}$ at $x = x_0$ i.e. $\left(\dfrac{dy}{dx}\right)_{x=x_0} = f'(x_0)$

$f'(x_0)$ represent the rate of change of y with respect to x at $x = x_0$.

Further, if two variables x and y are varying one with respect to another variable t i.e. if $y = f(t)$ and $x = g(t)$, then by chain rule.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \frac{dx}{dt} \neq 0$$

Hence, the rate of change y with respect to x can be calculated by using the rate of change of y and that of x both with respect to t.

Example 29.1 Find the rate of change of area of a circle with respect to its variable radius r, when $r = 3$ cm.

Solution : Let A be the area of a circle of radius r,

then $\quad A = \pi r^2$

$\therefore$ The rate of change of area A with respect to its radius r

$$\Rightarrow \quad \frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

when $r = 3$ cm, $\dfrac{dA}{dr} = 2\pi \times 3 = 6\pi$

Hence, the area of the circle is changing at the rate of 6π cm^2/cm

Example 29.2 A balloon which always remains spherical, has a variable diameter $\dfrac{3}{2}(2x+3)$. Determine the rate of change of volume with respect to x.

Solution : Radius (say r) of the spherical balloon $= \dfrac{1}{2}$ (diameter)

$$= \frac{1}{2} \times \frac{3}{2}(2x+3) = \frac{3}{4}(2x+3)$$

Let V be the volume of the balloon, then

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{3}{4}(2x+3)\right)^3$$

$$\Rightarrow \quad V = \frac{9}{16}\pi(2x+3)^3$$

$\therefore$ The rate of change of volume w.r. to 'x'

$$\frac{dV}{dx} = \frac{9}{16}\pi \times 3(2x+3)^2 \times 2 = \frac{27}{8}\pi(2x+3)^2$$

Hence, the volume is changing at the rate of $\frac{27}{8}\pi(2x+3)^2 \text{unit}^3/\text{unit}$

Example 29.3 A balloon which always remains spherical is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing, when its radius is 15 cm.

Solution : Let r be the radius of the spherical balloon and V be its volume at any time t, then

$$V = \frac{4}{3}\pi r^3$$

Diff. w.r. to 't' we get

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right).\frac{dr}{dt}$$

$$= \frac{4}{3}\pi.3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

But $\quad \frac{dV}{dt} = 900 \text{ cm}^3/\text{sec. (given)}$

So, $\quad 4\pi r^2 \frac{dr}{dt} = 900$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

when $r = 15$ cm,

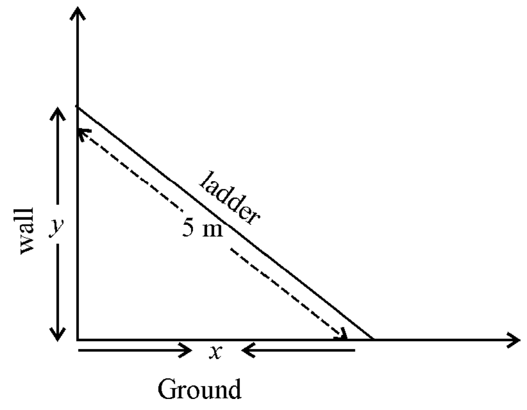
$$\frac{dr}{dt} = \frac{225}{\pi \times 15^2} = \frac{1}{\pi}$$

Hence, the radius of balloon is increasing at the rate of $\frac{1}{\pi}$ cm/sec , when its radius is 15 cm.

Example 29.4 A ladder 5 m long is leaning against a wall. The foot of the ladder is pulled along the ground, away from the wall, at the rate of 2m/sec. How fast is its height on the wall decreasing when the foot of ladder is 4m away from the wall?

Solution : Let the foot of the ladder be at a distance x metres from the wall and y metres be the height of the ladder at any time t, then

$$x^2 + y^2 = 25 \qquad ...(i)$$



Diff. w.r. to 't'. We get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \qquad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

But $$\frac{dx}{dt} = 2 \text{ m/sec. (given)}$$

$$\Rightarrow \qquad \frac{dy}{dt} = -\frac{x}{y} \times 2 = -\frac{2x}{y}$$

...(ii)

When $x = 4$m, from (i) $y^2 = 25 - 16 \Rightarrow y = 3$m

Putting $x = 4$m and $y = 3$m in (ii), we get

$$\frac{dy}{dx} = -\frac{2 \times 4}{3} = \frac{-8}{3}$$

Hence, the height of the ladder on the wall is decreasing at the rate of $\frac{8}{3}$ m/sec.

Example 29.5 The total revenue received from the sale of x units of a product is given by

$$R(x) = 10x^2 + 13x + 24$$

Find the marginal revenue when $x = 5$, where by marginal revenue we mean the rate of change of total revenue w.r. to the number of items sold at an instant.

Solution : Given $R(x) = 10x^2 + 13x + 24$

Since marginal revenue is the rate of change of the revenue with respect to the number of units sold, we have

$$\text{marginal revenue (MR)} = \frac{dR}{dx} = 20x + 13$$

when $x = 5$, MR $= 20 \times 5 + 13 = 113$

Hence, the marginal revenue = ` 113

Example 29.6 The total cost associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$$

Find the marginal cost when 17 units are produced, where by marginal cost we mean the instantaneous rate of change of the total cost at any level of output.

Solution : Given $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$

Since marginal cost is the rate of change of total cost w.r. to the output, we have

$$\text{Marginal Cost (MC)} = \frac{dC}{dx}$$

$$= 0.007 \times 3x^2 - 0.003 \times 2x + 15$$

$$= 0.021x^2 - 0.006x + 15$$

when x = 17,

$$MC = 0.021 \times 17^2 - 0.006 \times 17 + 15$$

$$= 6.069 - 0.102 + 15$$

$$= 20.967$$

Hence, marginal cost = ` 20.967

CHECK YOUR PROGRESS 29.1

1. The side of a square sheet is increasing at rate of 4 cm per minute. At what rate is the area increasing when the side is 8 cm long?
2. An edge of a variable cube is increasing at the rate of 3 cm per second. How fast is the volume of the cube increasing when the edge is 10 cm long.
3. Find the rate of change of the area of a circle with respect to its radius when the radius is 6 cm.
4. The radius of a spherical soap bubble is increasing at the rate of 0.2 cm/sec. Find the rate of increase of its surface area, when the radius is 7 cm.
5. Find the rate of change of the volume of a cube with respect to its edge when the edge is 5 cm.

29.2 APPROXIMATIONS

In this section, we shall give a meaning to the symbols dx and dy in such a way that the original meaning of the symbol $\frac{dy}{dx}$ coincides with the quotient when dy is divided by dx.

Let $y = f(x)$ be a function of x and Δx be a small change in x and let Δy be the corresponding change in y. Then,

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$$

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \varepsilon, \text{ where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

$$\Rightarrow \quad \Delta y = \frac{dy}{dx}\Delta x + \varepsilon \Delta x$$

$\because$ $\varepsilon \Delta x$ is a very-very small quantity that can be neglected, therefore

we have $\Delta y = \frac{dy}{dx}\Delta x$, approximately

This formula is very useful in the calculation of small change (or errors) in dependent variable corresponding to small change (or errors) in the independent variable.

SOME IMPORTANT TERMS

ABSOLUTE ERROR : The error Δx in x is called the absolute error in x.

RELATIVE ERROR : If Δx is an error in x, then $\frac{\Delta x}{x}$ is called relative error in x.

PERCENTAGE ERROR : If Δx is an error in x, then $\frac{\Delta x}{x} \times 100$ is called percentage error in x.

Note : We have $\Delta y = \frac{dy}{dx}\Delta x + \varepsilon.\Delta x$

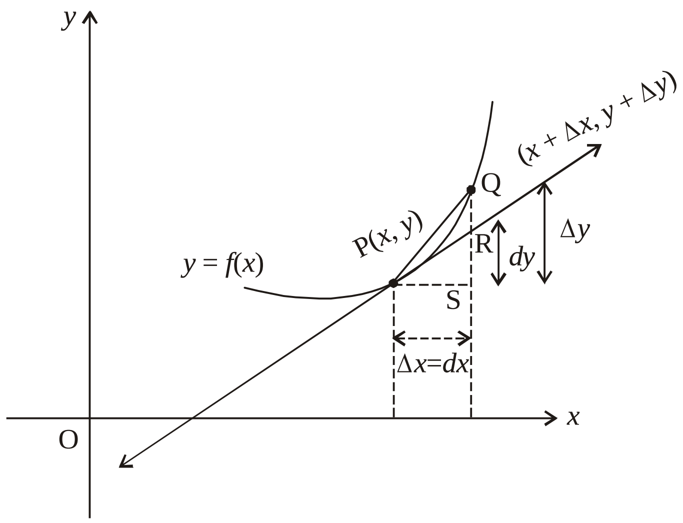
$\because$ $\varepsilon.\Delta x$ is very smal, therefore principal value of $\Delta y = \frac{dy}{dx}\Delta x$ which is called differential of y.

i.e. $\Delta y = \frac{dy}{dx}.\Delta x$

So, the differential of x is given by

$$dx = \frac{dx}{dx}.\Delta x = \Delta x$$

Hence, $dy = \frac{dy}{dx}dx$



To understand the geometrical meaning of dx, Δx, dy and Δy. Let us focus our attention to the portion of the graph of $y = f(x)$ in the neighbourhood of the point P(x, y) where a tangent can be drawn the curve. If Q$(x + \Delta x, y + \Delta y)$ be another point $(\Delta x \neq 0)$ on the curve, then the slope of line PQ will be $\frac{\Delta y}{\Delta x}$ which approaches the limiting value $\frac{dy}{dx}$ (slope of tangent at P).

Therefore, when $\Delta x \to 0$, Δy is approximately equal to dy.

Example 29.7 Using differentials, find the approximate value of $\sqrt{25.3}$

Solution : Let $y = \sqrt{x}$

Differentiating w.r.t. to 'x' we get

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Take $x = 25$ and $x + \Delta x = 25.3$, then $dx = \Delta x = 0.3$ when x = 25, $y = \sqrt{25} = 5$

$$\Delta y = \frac{dy}{dx}\Delta x = \frac{1}{2\sqrt{x}}\Delta x = \frac{1}{2\sqrt{25}} \times 0.3 = \frac{1}{10} \times 0.3 = 0.03$$

$\Rightarrow \Delta y = 0.03$ ($\because$ dy is approximately equal to Δy)

$$y + \Delta y = \sqrt{x + \Delta x} = \sqrt{25.3}$$

$\Rightarrow$ $\sqrt{25.3} = 5 + 0.03 = 5.03$ approximately

Example 29.8 Using differentials find the approximate value of $(127)^{\frac{1}{3}}$

Solution : Take $y = x^{\frac{1}{3}}$

Let $x = 125$ and $x + \Delta x = 127$, then $dx = \Delta x = 2$

When $x = 125$, $y = (125)^{\frac{1}{3}} = 5$

Now $\qquad y = x^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3x^{2/3}}$$

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3x^{2/3}}dx = \frac{1}{3(125)^{2/3}} \times 2 = \frac{2}{75}$$

$\Rightarrow \qquad \Delta y = \dfrac{2}{75}$

$(\because\ \Delta y = dy)$

Hence, $\qquad (127)^{\frac{1}{3}} = y + \Delta y = 5 + \dfrac{2}{75} = 5.026$ (Approximate)

Example 29.9 Find the approximate value of $f(3.02)$, where

$$f(x) = 3x^2 + 5x + 3$$

Solution : Let x = 3 and $x + \Delta x = 3.02$, then $dx = \Delta x = 0.02$

We have $\qquad f(x) = 3x^2 + 5x + 3$

when $x = 3$

$\Rightarrow \qquad f(3) = 3(3)^2 + 5(3) + 3 = 45$

Now $y = f(x)$

$\Rightarrow \qquad \Delta y = \dfrac{dy}{dx}\Delta x = (6x + 5)\Delta x$

$\Rightarrow \qquad \Delta y = (6 \times 3 + 5) \times 0.02 = 0.46$

$\therefore \quad f(3.02) = f(x + \Delta x) = y + \Delta y = 45 + 0.46 = 45.46$

Hence, the approximate value of $f(3.02)$ is 45.46.

Example 29.10 If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area.

Solution : Let r be the radius of the sphere and Δr be the error in measuring the radius. Then

$r = 9$ cm and $\Delta r = 0.03$ cm

Let S be the surface area of the sphere. Then

$S = 4\pi r^2$

$\Rightarrow \qquad \dfrac{dS}{dr} = 4\pi \times 2r = 8\pi r$

$$\left(\frac{dS}{dr}\right)_{\text{at } r=9} = 8\pi \times (9) = 72\pi$$

Let ΔS be the error in S, then

$$\Delta S = \frac{dS}{dr}\Delta r = 72\pi \times 0.03 = 2.16\pi \text{ cm}^2$$

Hence, approximate error in calculating the surface area is 2.16π cm^2.

Example 29.11 Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by 2%.

Solution : Let Δx be the change in x and ΔV be the corresponding change in V.

Given that $\frac{\Delta x}{x} \times 100 = 2 \Rightarrow \Delta x = \frac{2x}{100}$

we have $\quad V = x^3$

$\Rightarrow \quad \frac{dV}{dx} = 3x^2$

Now $\quad \Delta V = \frac{dV}{dx}\Delta x$

$\Rightarrow \quad \Delta V = 3x^2 \times \frac{2x}{100}$

$\Rightarrow \quad \Delta V = \frac{6}{100}.V$

Hence, the approximate change in volume is 6%.

CHECK YOUR PROGRESS 29.2

1. Using differentials, find the approximate value of $\sqrt{36.6}$.
2. Using differentials, find the appoximate value of $(25)^{\frac{1}{3}}$.
3. Using differentials, find the approximate value of $(15)^{\frac{1}{4}}$.
4. Using differentials, find the approximate value of $\sqrt{26}$.
5. If the radius of a sphere is measured as 7 m with an error of 0.02 m, find the approximate error in calculating its volume.
6. Find the percentage error in calculating the volume of a cubical box if an error of 1% is made in measuring the length of edges of the box.

29.3 SLOPE OF TANGENT AND NORMAL

Let $y = f(x)$ be a continuous curve and let $P(x_1, y_1)$ be a point on it then the slope PT at $P(x_1, y_1)$ is given by

$$\left(\frac{dy}{dx}\right) \text{at } (x_1, y_1) \qquad ....(i)$$

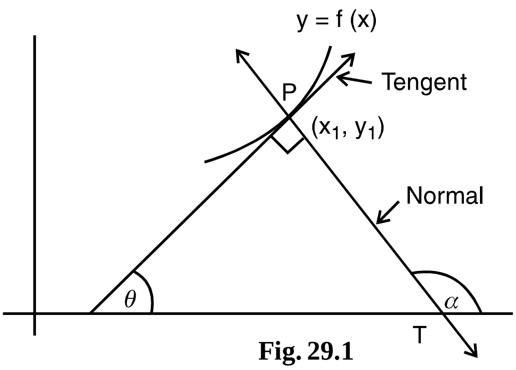


Fig. 29.1

and (i) is equal to $\tan\theta$

We know that a normal to a curve is a line perpendicular to the tangent at the point of contact

We know that $\alpha = \frac{\pi}{2} + \theta$ (From Fig. 10.1)

$$\Rightarrow \quad \tan\alpha = \tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$$

$$= -\frac{1}{\tan\theta}$$

$\therefore$ Slope of normal $= -\frac{1}{m} = \frac{-1}{\left(\frac{dy}{dx}\right)}$ at (x_1, y_1) or $-\left(\frac{dx}{dy}\right)$ at (x_1, y_1)

Note

1. The tangent to a curve at any point will be parallel to x-axis if $\theta = 0$, i.e, the derivative at the point will be zero.

   i.e. $\left(\frac{dx}{dy}\right)$ at $(x_1, y_1) = 0$

2. The tangent at a point to the curve $y = f(x)$ will be parallel to y-axis if $\frac{dy}{dx} = 0$ at that point.

Let us consider some examples :

Example 29.12 Find the slope of tangent and normal to the curve

$$x^2 + x^3 + 3xy + y^2 = 5 \text{ at } (1, 1)$$

Solution : The equation of the curve is

$$x^2 + x^3 + 3xy + y^2 = 5 \quad .....(i)$$

Differentialing (i),w.r.t. x, we get

$$2x + 3x^2 + 3\left[x\frac{dy}{dx} + y.1\right] + 2y\frac{dy}{dx} = 0 \quad .....(ii)$$

Substituting $x = 1$, $y = 1$, in (ii), we get

$$2\times1 + 3\times1 + 3\left[\frac{dy}{dx} + 1\right] + 2\frac{dy}{dx} = 0$$

or $\quad 5\frac{dy}{dx} = -8 \Rightarrow \quad \frac{dy}{dx} = -\frac{8}{5}$

$\therefore$ The slope of tangent to the curve at (1, 1) is $-\frac{8}{5}$

$\therefore$ The slope of normal to the curve at (1, 1) is $\frac{5}{8}$

Example 29.13 Show that the tangents to the curve $y = \frac{1}{6}\left[3x^5 + 2x^3 - 3x\right]$ at the points $x = \pm 3$ are parallel.

Solution : The equation of the curve is $y = \frac{3x^5 + 2x^3 - 3x}{6}$ .....(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\left(15x^4 + 6x^2 - 3\right)}{6}$$

$$\left(\frac{dy}{dx}\right)_{\text{at } x=3} = \frac{\left[15(3)^4 + 6(3)^2 - 3\right]}{6}$$

$$= \frac{1}{6}[15 \times 9 \times 9 + 54 - 3]$$

$$= \frac{3}{6}[405 + 17] = 211$$

$$\left(\frac{dy}{dx}\right) \text{at} x = -3 = \frac{1}{6}\left[15(-3)^4 + 6(-3)^2 - 3\right] = 211$$

$\therefore$ The tangents to the curve at $x = \pm 3$ are parallel as the slopes at $x = \pm 3$ are equal.

Example 29.14 The slope of the curve $6y^3 = px^2 + q$ at $(2, -2)$ is $\frac{1}{6}$.

Find the values of p and q.

Solution : The equation of the curve is

$$6y^3 = px^2 + q \quad .....(i)$$

Differentiating (i) w.r.t. x, we get

$$18y^2 \frac{dy}{dx} = 2px \quad .....(ii)$$

Putting $x = 2,\ y = -2,$ we get

$$18(-2)^2 \frac{dy}{dx} = 2p \cdot 2 = 4p$$

$$\therefore \qquad \frac{dy}{dx} = \frac{p}{18}$$

It is given equal to $\frac{1}{6}$

$$\therefore \qquad \frac{1}{6} = \frac{p}{18} \Rightarrow p = 3$$

$\therefore$ The equation of curve becomes

$$6y^3 = 3x^2 + q$$

Also, the point (2, –2) lies on the curve

$$\therefore \qquad 6(-2)^3 = 3(2)^2 + q$$

$$\Rightarrow \qquad -48 - 12 = q \quad \text{or} \quad q = -60$$

$\therefore$ The value of $p = 3,\ q = -60$

CHECK YOUR PROGRESS 29.3

1. Find the slopes of tangents and normals to each of the curves at the given points :

   (i) $y = x^3 - 2x$ at $x = 2$ (ii) $x^2 + 3y + y^2 = 5$ at (1, 1)

   (iii) $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$

2. Find the values of p and q if the slope of the tangent to the curve $xy + px + qy = 2$ at (1, 1) is 2.

3. Find the points on the curve $x^2 + y^2 = 18$ at which the tangents are parallel to the line x+y = 3.

4. At what points on the curve $y = x^2 - 4x + 5$ is the tangent perpendiculat to the line $2y + x - 7 = 0$.

29.4 EQUATIONS OF TANGENT AND NORMAL TO A CURVE

We know that the equation of a line passing through a point (x_1, y_1) and with slope m is

$$y - y_1 = m(x - x_1)$$

As discussed in the section before, the slope of tangent to the curve y = f (x) at (x_1, y_1) is given by $\left(\frac{dy}{dx}\right)$ at (x_1, y_1) and that of normal is $\left(-\frac{dx}{dy}\right)$ at (x_1, y_1)

$\therefore$ Equation of tangent to the curve y = f (x) at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} [x - x_1]$$

And, the equation of normal to the curve y = f(x) at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{-1}{\frac{dy}{dx}}\right)_{(x_1, y_1)} [x - x_1]$$

Note

(i) The equation of tangent to a curve is parallel to x-axis if $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$. In that case the equation of tangent is $y = y_1$.

(ii) In case $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} \to \infty$, the tangent at (x_1, y_1) is parallel to $y - \text{axis}$ and its equation is $x = x_1$

Let us take some examples and illustrate

Example 29.15 Find the equation of the tangent and normal to the circle $x^2 + y^2 = 25$ at the point (4, 3)

Solution : The equation of circle is

$$x^2 + y^2 = 25 \qquad \text{....(i)}$$

Differentialing (1), w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-x}{y}$$

$$\therefore \qquad \left(\frac{dy}{dx}\right)_{\text{at }(4,3)} = -\frac{4}{3}$$

$\therefore$ Equation of tangent to the circle at (4, 3) is

$$y - 3 = -\frac{4}{3}(x - 4)$$

or $\quad 4(x-4) + 3(y-3) = 0 \quad$ or, $\quad 4x + 3y = 25$

Also, slope of the normal $= \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(4,3)}} = \dfrac{3}{4}$

$\therefore$ Equation of the normal to the circle at (4,3) is

$$y - 3 = \frac{3}{4}(x - 4)$$

or $$4y - 12 = 3x - 12$$

$\Rightarrow$ $$3x = 4y$$

$\therefore$ Equation of the tangent to the circle at (4,3) is 4x+3y = 25

Equation of the normal to the circle at (4,3) is 3x = 4y

Example 29.16 Find the equation of the tangent and normal to the curve $16x^2 + 9y^2 = 144$ at the point (x_1, y_1) where $y_1 > 0$ and $x_1 = 2$

Solution : The equation of curve is

$$16x^2 + 9y^2 = 144 \qquad .....(i)$$

Differentiating (i), w.r.t. x we get

$$32x + 18y\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\frac{16x}{9y}$$

As $x_1 = 2$ and (x_1, y_1) lies on the curve

$\therefore$ $$16(2)^2 + 9(y^2) = 144$$

$\Rightarrow$ $$y^2 = \frac{80}{9} \Rightarrow y = \pm\frac{4}{3}\sqrt{5}$$

As $$y_1 > 0 \Rightarrow y = \frac{4}{3}\sqrt{5}$$

$\therefore$ Equation of the tangent to the curve at $\left(2, \frac{4}{3}\sqrt{5}\right)$ is

$$y - \frac{4}{3}\sqrt{5} = \left(-\frac{16x}{9y}\right)_{\text{at}\left(2, \frac{4\sqrt{5}}{3}\right)}[x - 2]$$

or $$y-\frac{4}{3}\sqrt{5}=-\frac{16}{9}\cdot\frac{2\times 3}{4\sqrt{5}}(x-2)$$ or $$y-\frac{4}{3}\sqrt{5}+\frac{8}{3\sqrt{5}}(x-2)=0$$

or $$3\sqrt{5}y-\frac{4}{3}\sqrt{5}\cdot 3\sqrt{5}+8(x-2)=0$$

$$3\sqrt{5}y-20+8x-16=0$$ or $$3\sqrt{5}y+8x=36$$

Also, equation of the normal to the curve at $\left(2,\frac{4}{3}\sqrt{5}\right)$ is

$$y-\frac{4}{3}\sqrt{5}=\left(\frac{9y}{16x}\right)_{\text{at}\left(2,\frac{4}{3}\sqrt{5}\right)}[x-2]$$

$$y-\frac{4}{3}\sqrt{5}=\frac{9}{16}\times\frac{2\sqrt{5}}{3}(x-2)$$

$$y-\frac{4}{3}\sqrt{5}=\frac{3\sqrt{5}}{8}(x-2)$$

$$3\times 8(y)-32\sqrt{5}=9\sqrt{5}(x-2)$$

$$24y-32\sqrt{5}=9\sqrt{5}\,x-18\sqrt{5}$$ or $$9\sqrt{5}x-24y+14\sqrt{5}=0$$

Example 29.17 Find the points on the curve $\frac{x^2}{9}-\frac{y^2}{16}=1$ at which the tangents are parallel to x-axis.

Solution : The equation of the curve is

$$\frac{x^2}{9}-\frac{y^2}{16}=1 \quad ....(i)$$

Differentiating (i) w.r.t. x we get

$$\frac{2x}{9}-\frac{2y}{16}\cdot\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=\frac{16x}{9y}$$

For tangent to be parallel to x-axis, $\frac{dy}{dx}=0$

$$\Rightarrow \quad \frac{16x}{9y}=0 \quad \Rightarrow \quad x=0$$

Putting x = 0 in (i), we get $y^2=-16 \;\; y=\pm 4i$

This implies that there are no real points at which the tangent to $\frac{x^2}{9}-\frac{y^2}{16}=1$ is parallel to x-axis.

Example 29.18 Find the equation of all lines having slope – 4 that are tangents to the curve $y=\frac{1}{x-1}$

Solution : $$y=\frac{1}{x-1} \qquad \text{.....(i)}$$

$\therefore$ $$\frac{dy}{dx}=-\frac{1}{(x-1)^2}$$

It is given equal to – 4

$\therefore$ $$\frac{-1}{(x-1)^2}=-4$$

$\Rightarrow$ $$(x-1)^2=\frac{1}{4}, \quad \Rightarrow \quad x=1\pm\frac{1}{2}\Rightarrow x=\frac{3}{2},\frac{1}{2}$$

Substituting $x=\frac{1}{2}$ in (i), we get

$$y=\frac{1}{\frac{1}{2}-1}=\frac{1}{-\frac{1}{2}}=-2$$

When $$x=\frac{3}{2}, \quad y=2$$

$\therefore$ The points are $\left(\frac{3}{2},2\right),\left(\frac{1}{2},-2\right)$

$\therefore$ The equations of tangents are

(a) $$y-2=-4\left(x-\frac{3}{2}\right), \quad \Rightarrow \; y-2=-4x+6 \quad \text{or} \quad 4x+y=8$$

(b) $$y+2=-4\left(x-\frac{1}{2}\right)$$

$\Rightarrow$ $$y+2=-4x+2 \quad \text{or} \quad 4x+y=0$$

Example 29.19 Find the equation of the normal to the curve $y=x^3$ at (2 , 8)

Solution : $$y=x^3 \qquad \Rightarrow \frac{dy}{dx}=3x^2$$

$\therefore$ $$\left(\frac{dy}{dx}\right)_{at x=2}=12$$

$\therefore$ Slope of the normal $= -\dfrac{1}{12}$

$\therefore$ Equation of the normal is

$$y - 8 = -\frac{1}{12}(x-2)$$

or $\quad 12(y-8)+(x-2)=0 \quad$ or $\quad x+12y=98$

CHECK YOUR PROGRESS 29.4

1. Find the equation of the tangent and normal at the indicated points :

   (i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

   (ii) $y = x^2$ at $(1, 1)$

   (iii) $y = x^3 - 3x + 2$ at the point whose x–coordinate is 3

2. Find the equation of the targent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at (x_1, y_1)

3. Find the equation of the tangent to the hyperbola

   $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \quad$ at (x_0, y_0)

4. Find the equation of normals to the curve

   $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$

5. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$

29.5 Mathematical formulation of Rolle's Theorem

Let f be a real function defined in the closed interval [a, b] such that

(i) f is continuous in the closed interval [a, b]

(ii) f is differentiable in the open inteval (a, b)

(iii) f (a) = f (b)

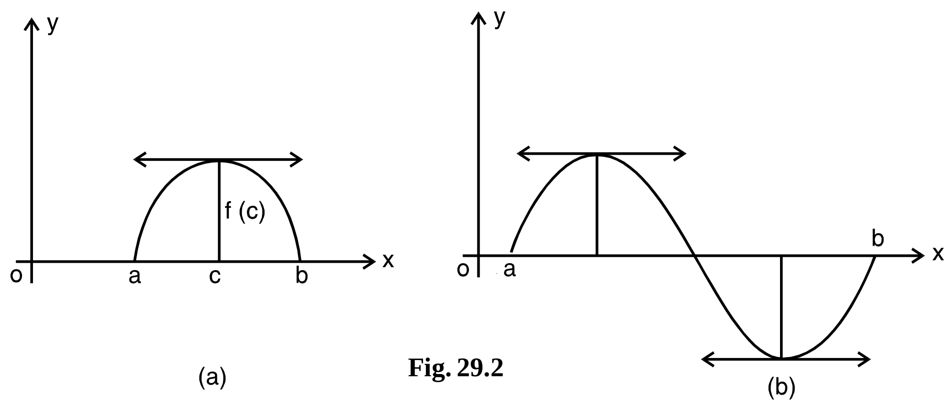


Fig. 29.2

Then there is at least one point c in the open inteval (a, b) such that $f'(c) = 0$

Remarks

(i) The remarks "at least one point" says that there can be more than one point $c \in (a, b)$ such that $f'(c) = 0$.

(ii) The condition of continuity of f on [a, b] is essential and can not be relaxed

(iii) The condition of differentiability of f on (a,b) is also essential and can not be relaxed.

For example $f(x) = |x|, x \in [-1,1]$ is continuous on [–1,1] and differentiable on (–1, 1) and Rolle's Theorem is valid for this

Let us take some examples

Example 29.20 Verify Rolle's for the function

$$f(x) = x(x-1)(x-2), x \in [0, 2]$$

Solution : $$f(x) = x(x-1)(x-2)$$

$$= x^3 - 3x^2 + 2x$$

(i) f (x) is a polynomial function and hence continuous in [0, 2]

(ii) f (x) is differentiable on (0,2)

(iii) Also f (0) = 0 and f (2) = 0

$\therefore$ $$f(0) = f(2)$$

$\therefore$ All the conditions of Rolle's theorem are satisfied.

Also, $$f'(x) = 3x^2 - 6x + 2$$

$\therefore f'(c) = 0$ gives $3c^2 - 6c + 2 = 0$ $\Rightarrow$ $c = \dfrac{6 \pm \sqrt{36-24}}{6}$

$\Rightarrow$ $$c = 1 \pm \frac{1}{\sqrt{3}}$$

We see that both the values of c lie in (0, 2)

Example 29.21 Discuss the applicability of Rolle's Theorem for

$$f(x) = \sin x - \sin 2x,\ x \in [0, \pi] \qquad ....(i)$$

(i) is a sine function. It is continuous and differentiable on $(0, \pi)$

Again, we have, $f(0) = 0$ and $f(\pi) = 0$

$$\Rightarrow \qquad f(\pi) = f(0) = 0$$

$\therefore$ All the conditions of Rolle's theorem are satisfied

Now $\qquad f'(c) = 2\left[2\cos^2 c - 1\right] - \cos c = 0$

or $\qquad 4\cos^2 c - \cos c - 2 = 0$

$$\therefore \qquad \cos c = \frac{1 \pm \sqrt{1+32}}{8}$$

$$= \frac{1 \pm \sqrt{33}}{8}.$$

As $\sqrt{33} < 6$

$$\therefore \qquad \cos c < \frac{7}{8} = 0.875$$

which shows that c lies between 0 and π

CHECK YOUR PROGRESS 29.5

Verify Rolle's Theorem for each of the following functions :

(i) $f(x) = \frac{x^3}{3} - \frac{5x^2}{3} + 2x,\ x \in [0, 3]$

(ii) $f(x) = x^2 - 1$ on $[-1, 1]$

(iii) $f(x) = \sin x + \cos x - 1$ on $\left(0, \frac{\pi}{2}\right)$

(iv) $f(x) = \left(x^2 - 1\right)\left(x - 2\right)$ on $[-1, 2]$

29.6 LANGRANGE'S MEAN VALUE THEOREM

This theorem improves the result of Rolle's Theorem saying that it is not necessary that tangent may be parallel to x-axis. This theorem says that the tangent is parallel to the line joining the end points of the curve. In other words, this theorem says that there always exists a point on the graph, where the tangent is parallel to the line joining the end-points of the graph.

29.6.1 Mathematical Formulation of the Theorem

Let f be a real valued function defined on the closed interval [a, b] such that

(a) f is continuous on [a, b], and

(b) f is differentiable in (a, b)

(c) $f(b) \neq f(a)$

then there exists a point c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Remarks

When f (b) = f (a), f '(c) = 0 and the theorem reduces to Rolle's Theorem

Let us consider some examples

Example 29.22 Verify Langrange's Mean value theorem for

$$f(x) = (x-3)(x-6)(x-9) \text{ on } [3, 5]$$

Solution : $f(x) = (x-3)(x-6)(x-9)$

$$= (x-3)(x^2 - 15x + 54)$$

or $$f(x) = x^3 - 18x^2 + 99x - 162 \qquad ...(i)$$

(i) is a polynomial function and hence continuous and differentiable in the given interval

Here, $f(3) = 0,\ f(5) = (2)(-1)(-4) = 8$

$\therefore$ $f(3) \neq f(5)$

$\therefore$ All the conditions of Mean value Theorem are satisfied

$\therefore$ $$f'(c) = \frac{f(5) - f(3)}{5 - 3} = \frac{8 - 0}{2} = 4$$

Now $f'(x) = 3x^2 - 36x + 99$

$\therefore$ $3c^2 - 36c + 99 = 4$ or $3c^2 - 36c + 95 = 0$

$\therefore$ $$c = \frac{36 \pm \sqrt{1296 - 1140}}{6} = \frac{36 \pm 12.5}{6}$$

$= 8.08$ or 3.9

$c = 3.9 \in (3, 5)$

$\therefore$ Langranges mean value theorem is verified

Example 29.23 Find a point on the parabola $y = (x-4)^2$ where the tangent is parallel to the chord joining $(4, 0)$ and $(5, 1)$

Solution : Slope of the tangent to the given curve at any point is given by $(f'(x))$ at that point.

$$f'(x) = 2(x-4)$$

Slope of the chord joining $(4, 0)$ and $(5, 1)$ is

$$\frac{1-0}{5-4}=1 \qquad \left[\because m=\frac{y_2-y_1}{x_2-x_1}\right]$$

$\therefore$ According to mean value theorem

$$2(x-4)=1 \qquad \text{or} \qquad (x-4)=\frac{1}{2}$$

$$\Rightarrow \qquad x=\frac{9}{2}$$

which lies between 4 and 5

Now $\qquad y=(x-4)^2$

When $\qquad x=\frac{9}{2},\ y=\left(\frac{9}{2}-4\right)^2=\frac{1}{4}$

$\therefore$ The required point is $\left(\frac{9}{2},\frac{1}{4}\right)$

CHECK YOUR PROGRESS 29.6

1. Check the applicability of Mean Value Theorem for each of the following functions :

   (i) $f(x)=3x^2-4$ on [2, 3]

   (ii) $f(x)=\log x$ on [1, 2]

   (iii) $f(x)=x+\frac{1}{x}$ on [1, 3]

   (iv) $f(x)=x^3-2x^2-x+3$ on [0, 1]

2. Find a point on the parabola $y=(x+3)^2$, where the tangent is parallel to the chord joining $(3,0)$ and $(-4,1)$

29.7 INCREASING AND DECREASING FUNCTIONS

You have already seen the common trends of an increasing or a decreasing function. Here we will try to establish the condition for a function to be an increasing or a decreasing.

Let a function $f(x)$ be defined over the closed interval [a,b].

Let $x_1, x_2 \in [a,b]$, then the function f(x) is said to be an increasing function in the given interval if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$. It is said to be strictly increasing if $f(x_2) > f(x_1)$ for all $x_2 > x_1$, $x_1, x_2 \in [a,b]$.

In Fig. 29.3, sin x increases from -1 to +1 as x increases from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

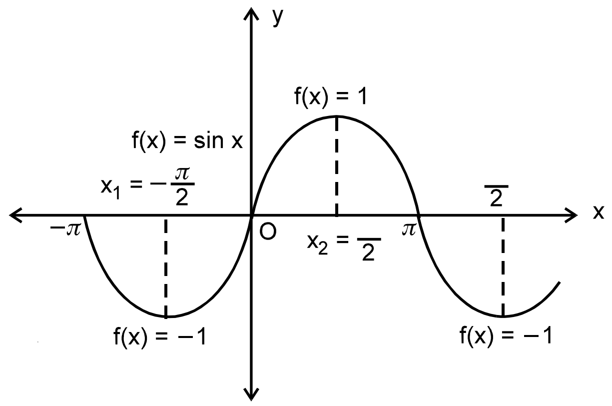


Fig. 29. 3

Note : A function is said to be an increasing function in an interval if $f(x+h) > f(x)$ for all x belonging to the interval when h is positive.

A function f(x) defined over the closed interval [a, b] is said to be a decreasing function in the given interval, if $f(x_2) \leq f(x_1)$, whenever $x_2 > x_1$, $x_1, x_2 \in [a,b]$. It is said to be strictly decreasing if $f(x_1) > f(x_2)$ for all $x_2 > x_1$, $x_1, x_2 \in [a,b]$.

In Fig. 29.4, cos x decreases from 1 to −1 as x increases from 0 to π.

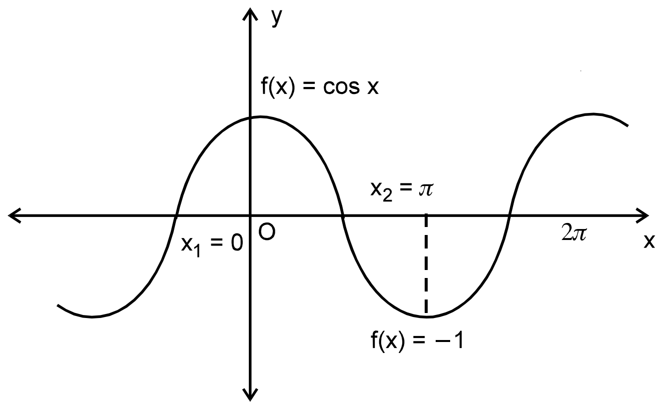


Fig. 29.4

Note : A function is said to be a decreasing in an internal if $f(x+h) < f(x)$ for all x belonging to the interval when h is positive.

29.7.1 MONOTONIC FUNCTIONS

Let x_1, x_2 be any two points such that $x_1 < x_2$ in the interval of definition of a function f(x). Then a function f(x) is said to be monotonic if it is either increasing or decreasing. It is said to be monotonically increasing if $f(x_2) \geq f(x_1)$ for all $x_2 > x_1$ belonging to the interval and monotonically decreasing if $f(x_1) \geq f(x_2)$.

Example 29.24 Prove that the function $f(x) = 4x + 7$ is monotonic for all values of $x \in R$.

Solution : Consider two values of x say $x_1, x_2 \in R$

such that $$x_2 > x_1 \qquad .....(1)$$

Multiplying both sides of (1) by 4, we have $4x_2 > 4x_1$ .....(2)

Adding 7 to both sides of (2), to get

$$4x_2 + 7 > 4x_1 + 7$$

We have $$f(x_2) > f(x_1)$$

Thus, we find $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

Hence the given function $f(x) = 4x + 7$ is monotonic function. (monotonically increasing).

Example 29.25 Show that

$$f(x) = x^2$$

is a strictly decreasing function for all $x < 0$.

Solution : Consider any two values of x say x_1, x_2 such that

$$x_2 > x_1, \qquad x_1, x_2 < 0 \qquad ....(i)$$

Order of the inequality reverses when it is multiplied by a negative number. Now multiplying (i) by x_2, we have

$$x_2 \cdot x_2 < x_1 \cdot x_2$$

or, $$x_2^2 < x_1 x_2 \qquad ......(ii)$$

Now multiplying (i) by x_1, we have

$$x_1 \cdot x_2 < x_1 \cdot x_1$$

or, $$x_1 x_2 < x_1^2 \qquad .....(iii)$$

From (ii) and (iii), we have

$$x_2^2 < x_1 x_2 < x_1^2$$

or, $$x_2^2 < x_1^2$$

or, $\qquad f(x_2) < f(x_1)$ .....(iv)

Thus, from (i) and (iv), we have for

$$x_2 > x_1,$$

$$f(x_2) < f(x_1)$$

Hence, the given function is strictly decreasing for all $x < 0$.

CHECK YOUR PROGRESS 29.7

1. (a) Prove that the function

$$f(x) = 3x + 4$$

is monotonic increasing function for all values of $x \in R$.

(b) Show that the function

$$f(x) = 7 - 2x$$

is monotonically decreasing function for all values of $x \in R$.

(c) Prove that $f(x) = ax + b$ where a, b are constants and $a > 0$ is a strictly increasing function for all real values of x.

2. (a) Show that $f(x) = x^2$ is a strictly increasing function for all real $x > 0$.

(b) Prove that the function $f(x) = x^2 - 4$ is monotonically increasing for $x > 2$ and monotonically decreasing for $-2 < x < 2$ where $x \in R$.

Theorem 1 : If f(x) is an increasing function on an open interval]a, b[, then its derivative f '(x) is positive at this point for all $x \in [a, b]$.

Proof : Let (x, y) or [x, f (x)] be a point on the curve y = f (x)

For a positive δx, we have

$$x + \delta x > x$$

Now, function f(x) is an increasing function

$\therefore \qquad f(x + \delta x) > f(x)$

or, $\qquad f(x + \delta x) - f(x) > 0$

or, $\qquad \dfrac{f(x + \delta x) - f(x)}{\delta x} > 0 \quad [\because \delta x > 0]$

Taking δx as a small positive number and proceeding to limit, when $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} > 0$$

or, $\qquad f'(x) > 0$

Thus, if $y = f(x)$ is an increasing function at a point, then f '(x) is positive at that point.

Theorem 2 : If f (x) is a decreasing function on an open interval]a, b[then its derivative f '(x) is negative at that point for all $x \in [a, b]$.

Proof : Let (x, y) or [x, f(x)] be a point on the curve y = f(x)

For a positive δx, we have $x + \delta x > x$

Since the function is a decreasing function

$\therefore$ $f(x+\delta x) < f(x)$ $\delta x > 0$

or, $f(x+\delta x) - f(x) < 0$

Dividing by δx, we have $\dfrac{f(x+\delta x) - f(x)}{\delta x} < 0 \quad \delta x > 0$

or, $\lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x} < 0$

or, $f'(x) < 0$

Thus, if $y = f(x)$ is a decreasing function at a point, then, $f'(x)$ is negative at that point.

Note : If $f(x)$ is derivable in the closed interval [a,b], then f (x) is

(i) increasing over [a,b], if f '(x)>0 in the open interval]a,b[

(ii) decreasing over [a,b], if f '(x)<0 in the open interval]a,b[.

29.8 RELATION BETWEEN THE SIGN OF THE DERIVATIVE AND MONOTONICITY OF FUNCTION

Consider a function whose curve is shown in the Fig. 29.5

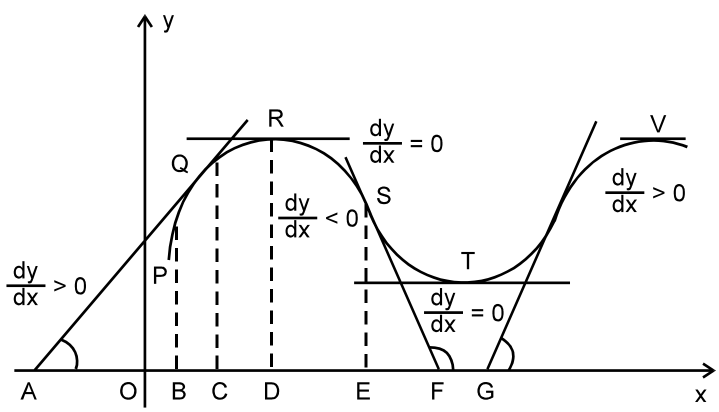


Fig. 29.5

We divide, our study of relation between sign of derivative of a function and its increasing or decreasing nature (monotonicity) into various parts as per Fig. 29.5

(i) P to R (ii) R to T (iii) T to V

(i) We observe that the ordinate (y-coordinate) for every succeeding point of the curve from P to R increases as also its x-coordinate. If (x_2, y_2) are the coordinates of a point that succeeds (x_1, y_1) then $x_2 > x_1$ yields $y_2 > y_1$ or $f(x_2) > f(x_1)$.

Also the tangent at every point of the curve between P and R makes acute angle with the positive direction of x-axis and thus the slope of the tangent at such points of the curve (except at R) is positive. At R where the ordinate is maximum the tangent is parallel to x-axis, as a result the slope of the tangent at R is zero.

We conclude for this part of the curve that

(a) The function is monotonically increasing from P to R.

(b) The tangent at every point (except at R) makes an acute angle with positive direction of x-axis.

(c) The slope of tangent is positive i.e. $\frac{dy}{dx} > 0$ for all points of the curve for which y is increasing.

(d) The slope of tangent at R is zero i.e. $\frac{dy}{dx} = 0$ where y is maximum.

(ii) The ordinate for every point between R and T of the curve decreases though its x-coordinate increases. Thus, for any point $x_2 > x_1$ yelds $y_2 < y_1$, or $f(x_2) < f(x_1)$.

Also the tangent at every point succeeding R along the curve makes obtuse angle with positive direction of x-axis. Consequently, the slope of the tangent is negative for all such points whose ordinate is decreasing. At T the ordinate attains minimum value and the tangent is parallel to x-axis and as a result the slope of the tangent at T is zero.

We now conclude :

(a) The function is monotonically decreasing from Rto T.

(b) The tangent at every point, except at T, makes obtuse angle with positive direction of x-axis.

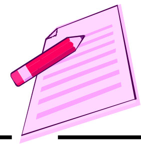
(c) The slope of the tangent is negative i.e., $\frac{dy}{dx} < 0$ for all points of the curve for which y is decreasing.

(d) The slope of the tangent at T is zero i.e. $\frac{dy}{dx} = 0$ where the ordinate is minimum.

(iii) Again, for every point from T to V

The ordinate is constantly increasing, the tangent at every point of the curve between T and V makes acute angle with positive direction of x-axis. As a result of which the slope of the tangent at each of such points of the curve is positive.

Conclusively,



$$\frac{dy}{dx} > 0$$

at all such points of the curve except at T and V, where $\frac{dy}{dx} = 0$. The derivative $\frac{dy}{dx} < 0$ on one side, $\frac{dy}{dx} > 0$ on the other side of points R, T and V of the curve where $\frac{dy}{dx} = 0$.

Example 29.26 Find for what values of x, the function

$$f(x) = x^2 - 6x + 8$$

is increasing and for what values of x it is decreasing.

Solution : $f(x) = x^2 - 6x + 8$

$$f'(x) = 2x - 6$$

For $f(x)$ to be increasing, $f'(x) > 0$

i.e., $2x - 6 > 0$ or, $2(x-3) > 0$

or, $x - 3 > 0$ or, $x > 3$

The function increases for x>3.

For $f(x)$ to be decreasing,

$$f'(x) < 0$$

i.e., $2x - 6 < 0$ or, $x - 3 < 0$

or, $x < 3$

Thus, the function decreases for $x < 3$.

Example 29.27 Find the interval in which $f(x) = 2x^3 - 3x^2 - 12x + 6$ is increasing or decreasing.

Solution : $f(x) = 2x^3 - 3x^2 - 12x + 6$

$$f'(x) = 6x^2 - 6x - 12$$

$$= 6(x^2 - x - 2)$$

$$= 6(x-2)(x+1)$$

For $f(x)$ to be increasing function of x,

$$f'(x) > 0$$

i.e. $6(x-2)(x+1) > 0$ or, $(x-2)(x+1) > 0$

Since the product of two factors is positive, this implies either both are positive or both are negative.

Either $x-2>0$ and $x+1>0$ or $x-2<0$ and $x+1<0$

i.e. $x>2$ and $x>-1$ | i.e. $x<2$ and $x<-1$

$x>2$ implies $x>-1$ | $x<-1$ implies $x<2$

$\therefore$ $x>2$ | $\therefore$ $x<-1$

Hence f (x) is increasing for $x > 2$ or $x < -1$.

Now, for f (x) to be decreasing,

$$f'(x)<0$$

or, $6(x-2)(x+1)<0$ or, $(x-2)(x+1)<0$

Since the product of two factors is negative, only one of them can be negative, the other positive.

Therefore,

Either | or

$x-2>0$ and $x+1<0$ | $x-2<0$ and $x+1>0$

i.e. $x>2$ and $x<-1$ | i.e. $x<2$ and $x>-1$

There is no such possibility that $x>2$ and at the same time $x<-1$ | This can be put in this form $-1<x<2$

$\therefore$ The function is decreasing in $-1<x<2$.

Example 29.28 Determine the intervals for which the function

$$f(x)=\frac{x}{x^2+1}$$ is increasing or decreasing.

Solution :

$$f'(x)=\frac{\left(x^2+1\right)\frac{dx}{dx}-x\cdot\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}$$

$$=\frac{\left(x^2+1\right)-x\cdot(2x)}{\left(x^2+1\right)^2}$$

$$=\frac{1-x^2}{\left(x^2+1\right)^2}$$

$$\therefore \quad f'x=\frac{(1-x)(1+x)}{\left(x^2+1\right)^2}$$

As $\left(x^2+1\right)^2$ is positive for all real x.

Therefore, if $-1 < x < 0, (1-x)$ is positive and $(1+x)$ is positive, so $f'(x) > 0$;

$\therefore$ If $\quad 0 < x < 1, (1-x)$ is positive and $(1+x)$ is positive, so f'$f'(x) > 0$;

If $\quad x < -1, (1-x)$ is positive and $(1+x)$ is negative, so $f'(x) < 0$;

$x > 1, (1-x)$ is negative and $(1+x)$ is positive, so $f'(x) < 0$;

Thus we conclude that

the function is increasing $\quad$ for $-1 < x < 0$ and $0 < x < 1$

or, $\quad$ for $-1 < x < 1$

and the function is decreasing for $x < -1$ or $x > 1$

Note : Points where $f'(x) = 0$ are critical points. Here, critical points are $x = -1$, $x = 1$.

Example 29.29 Show that

(a) $\quad f(x) = \cos x$ is decreasing in the interval $0 \le x \le \pi$.

(b) $\quad f(x) = x - \cos x$ is increasing for all x.

Solution :(a) $\quad f(x) = \cos x$

$$f'(x) = -\sin x$$

$f(x)$ is decreasing

If $\quad f'(x) < 0$

i.e., $\quad -\sin x < 0$

i.e., $\quad \sin x > 0$

sin x is positive in the first quadrant and in the second quadrant, therefore, sin x is positive in $0 \le x \le \pi$

$\therefore \quad f(x)$ is decreasing in $0 \le x \le \pi$

(b) $\quad f(x) = x - \cos x$

$$f'(x) = 1 - (-\sin x)$$

$$= 1 + \sin x$$

Now, we know that the minimum value of sinx is –1 and its maximum; value is 1 i.e.,sin x lies between –1 and 1 for all x,

i.e., $\quad -1 \le \sin x \le 1 \quad$ or $\quad 1-1 \le 1+\sin x \le 1+1$

or $\quad 0 \le 1 + \sin x \le 2$

or $\quad 0 \le f'(x) \le 2$

or $\quad 0 \le f'(x)$

or $\quad f'(x) \geq 0$

$\Rightarrow f(x) = x - \cos x$ is increasing for all x.

CHECK YOUR PROGRESS 29.8

Find the intervals for which the followiong functions are increasing or decreasing.

1. (a) $f(x) = x^2 - 7x + 10$ (b) $f(x) = 3x^2 - 15x + 10$

2. (a) $f(x) = x^3 - 6x^2 - 36x + 7$ (b) $f(x) = x^3 - 9x^2 + 24x + 12$

3. (a) $y = -3x^2 - 12x + 8$ (b) $f(x) = 1 - 12x - 9x^2 - 2x^3$

4. (a) $y = \frac{x-2}{x+1}, x \neq -1$ (b) $y = \frac{x^2}{x-1}, x \neq 1$ (c) $y = \frac{x}{2} + \frac{2}{x}, x \neq 0$

5. (a) Prove that the function log sin x is decreasing in $\left[\frac{\pi}{2}, \pi\right]$

   (b) Prove that the function cos x is increasing in the interval $[\pi, 2\pi]$

   (c) Find the intervals in which the function $\cos\left(2x + \frac{\pi}{4}\right), 0 \leq x \leq \pi$ is decreasing or increasing.

Find also the points on the graph of the function at which the tangents are parallel to x-axis.

29.9 MAXIMUM AND MINIMUM VALUES OF A FUNCTION

We have seen the graph of a continuous function. It increases and decreases alternatively. If the value of a continious function increases upto a certain point then begins to decrease, then this point is called point of maximum and corresponding value at that point is called maximum value of the function. A stage comes when it again changes from decreasing to increasing . If the value of a continuous function decreases to a certain point and then begins to increase, then value at that point is called minimum value of the function and the point is called point of minimum.

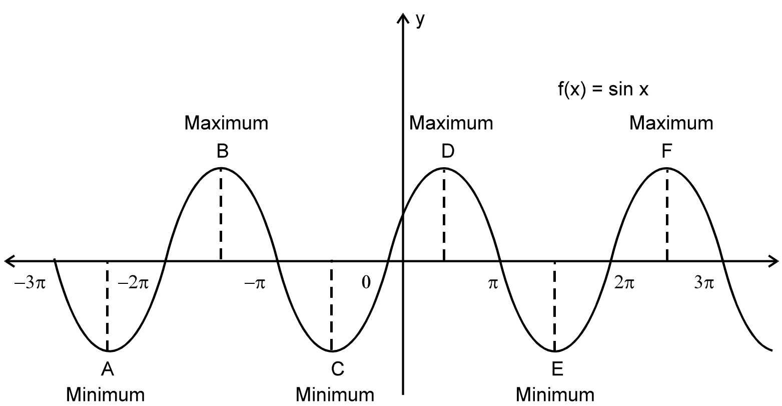


Fig. 29.6

Fig. 29.6 shows that a function may have more than one maximum or minimum values. So, for continuous function we have maximum (minimum) value in an interval and these values are not absolute maximum (minimum) of the function. For this reason, we sometimes call them as local maxima or local minima.

A function f (x) is said to have a maximum or a local maximum at the point x = a where a – b < a < a + b (See Fig. 29.7), if $f(a) \geq f(a \pm b)$ for all sufficiently small positive b.

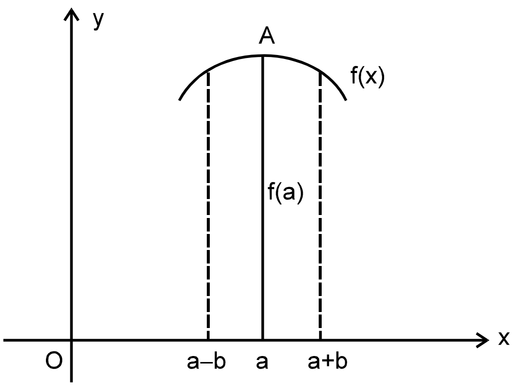


Fig. 29.7

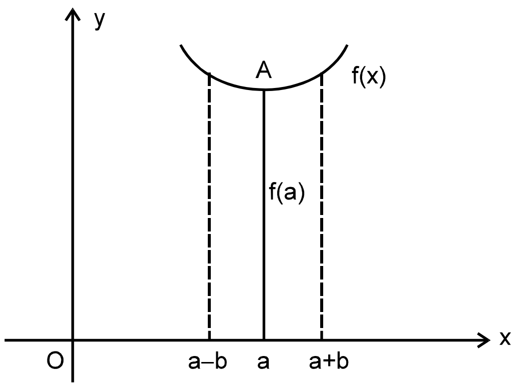


Fig. 29.8

A maximum (or local maximum) value of a function is the one which is greater than all other values on either side of the point in the immediate neighbourhood of the point.

A function f (x) is said to have a minimum (or local minimum) at the point x = a if $f(a) \leq f(a \pm b)$ where a – b< a < a + b

for all sufficiently small positive b.

In Fig. 25.8, the function f(x) has local minimum at the point x = a.

A minimum (or local miunimum) value of a function is the one which is less than all other values, on either side of the point in the immediate neighbourhood of the point.

Note : A neighbourhood of a point $x \in R$ is defined by open internal $]x-\in[$, when $\in > 0$.

29.9.1 CONDITIONS FOR MAXIMUM OR MINIMUM

We know that derivative of a function is positive when the function is increasing and the derivative is negative when the function is decreasing. We shall apply this result to find the condition for maximum or a function to have a minimum. Refer to Fig. 25.6, points B,D, F are points of maxima and points A,C,E are points of minima.

Now, on the left of B, the function is increasing and so $f'(x) > 0$, but on the right of B, the function is decreasing and, therefore, $f'(x) < 0$. This can be achieved only when f '(x) becomes zero somewhere in betwen. We can rewrite this as follows :

A function f (x) has a maximum value at a point if (i) f '(x) = 0 and (ii) f '(x) changes sign from positive to negative in the neighbourhood of the point at which f '(x)=0 (points taken from left to right).

Now, on the left of C (See Fig. 29.6), function is decreasing and f '(x) therefore, is negative and on the right of C, f (x) is increasing and so f '(x) is positive. Once again f '(x) will be zero before having positive values. We rewrite this as follows :

A function f (x) has a minimum value at a point if (i) f '(x)=0, and (ii) f '(x) changes sign from negative to positive in the neighbourhood of the point at which f '(x) = 0.

We should note here that f '(x) = 0 is necessary condition and is not a sufficient condition for maxima or minima to exist. We can have a function which is increasing, then constant and then again increasing function. In this case, f '(x) does not change sign. The value for which f '(x) = 0 is not a point of maxima or minima. Such point is called point of inflexion.

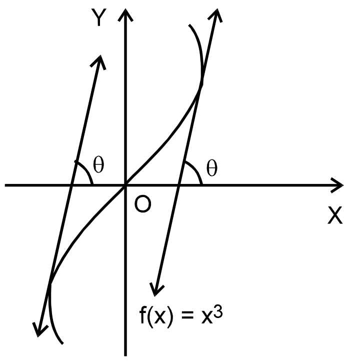


Fig. 29.9

For example, for the function $f(x) = x^3, x = 0$ is the point of inflexion as $f'(x) = 3x^2$ does not change sign as x passes through 0. f'(x) is positive on both sides of the value '0' (tangents make acute angles with x-axis) (See Fig. 29.9).

Hence $f(x) = x^3$ has a point of inflexion at x = 0.

The points where f '(x) = 0 are called stationary points as the rate ofchange of the function is zero there. Thus points of maxima and minima are stationary points.

Remarks

The stationary points at which the function attains either local maximum or local minimum values are also called extreme points and both local maximum and local minimum values are called extreme values of f (x). Thus a function attains an extreme value at x=a if f (a) is either a local maximum or a local minimum.

29.9.2 METHOD OF FINDING MAXIMA OR MINIMA

We have arrived at the method of finding the maxima or minima of a function. It is as follows :

(i) Find f'(x)

(ii) Put f '(x)=0 and find stationary points

(iii) Consider the sign of f '(x) in the neighbourhood of stationary points. If it changes sign from +ve to –ve, then f (x) has maximum value at that point and if f '(x) changes sign from –ve to +ve, then f (x) has minimum value at that point.

(iv) If f '(x) does not change sign in the neighbourhood of a point then it is a point of inflexion.

Example 29.30 Find the maximum (local maximum) and minimum (local minimum) points of the function $f(x) = x^3 - 3x^2 - 9x$.

Solution : Here

$$f(x) = x^3 - 3x^2 - 9x$$

$$f'(x) = 3x^2 - 6x - 9$$

Step I. Now $\quad f'(x) = 0$ gives us $3x^2 - 6x - 9 = 0$

or $\quad x^2 - 2x - 3 = 0$

or $\quad (x-3)(x+1) = 0$

or $\quad x = 3, -1$

$\therefore$ Stationary points are $\quad x = 3, x = -1$

Step II. At $\quad x = 3$

For $\quad x < 3 \quad f'(x) < 0$

and for $\quad x > 3 \quad f'(x) > 0$

$\therefore$ f '(x) changes sign from –ve to +ve in the neighbourhood of 3.

$\therefore$ f (x) has minimum value at x = 3.

Step III. At $\quad x = -1,$

For $\quad x < -1, \quad f'(x) > 0$

and for $\quad x > -1, \quad f'(x) < 0$

$\therefore$ f '(x) changes sign from +ve to –ve in the neighbourhood of –1.

$\therefore$ f(x) has maximum value at x=–1.

$\therefore$ x = –1 and x = 3 give us points of maxima and minima respectively. If we want to find maximum value (minimum value), then we have

$$\text{maximum value} = f(-1) = (-1)^3 - 3(-1)^2 - 9(-1)$$

$$= -1 - 3 + 9 = 5$$

and $\quad$ $$\text{minimum value} = f(3) = 3^3 - 3(3)^2 - 9(3) = -27$$

$\therefore$ $(-1, 5)$ and $(3, -27)$ are points of local maxima and local minima respectively.

Example 29.31 Find the local maximum and the local minimum of the function

$$f(x) = x^2 - 4x$$

Solution : $\quad f(x) = x^2 - 4x$

$\therefore \quad f'(x) = 2x - 4 = 2(x-2)$

Putting f ' (x)=0 yields 2x–4=0, i.e., x=2.

We have to examine whether x = 2 is the point of local maximum or local minimum or neither maximum nor minimum.

Let us take x = 1.9 which is to the left of 2 and x = 2.1 which is to the right of 2 and find f'(x) at these points.

Notes

$$f'(1.9) = 2(1.9-2) < 0$$

$$f'(2.1) = 2(2.1-2) > 0$$

Since $f'(x) < 0$ as we approach 2 from the left and $f'(x) > 0$ as we approach 2 from the right, therefore, there is a local minimum at x = 2.

We can put our findings for sign of derivatives of f(x) in any tabular form including the one given below :

sign of f ' (x)

point x = 2

left of 2	right of 2
f ' (x) < 0	f ' (x) < 0

Local minimum

Example 29.32 Find all local maxima and local minima of the function

$$f(x) = 2x^3 - 3x^2 - 12x + 8$$

Solution : $$f(x) = 2x^3 - 3x^2 - 12x + 8$$

$\therefore$ $$f'(x) = 6x^2 - 6x - 12$$

$$= 6(x^2 - x - 2)$$

$\therefore$ $$f'(x) = 6(x+1)(x-2)$$

Now solving f '(x)=0 for x, we get

$$6(x+1)(x-2) = 0$$

$\Rightarrow$ $$x = -1, 2$$

Thus $f'(x) = 0$ at $x = -1, 2.$

We examine whether these points are points of local maximum or local minimum or neither of them.

Consider the point x = – 1

Let us take x = –1.1 which is to the left of –1 and x = – 0.9 which is to the right of -1 and find f ' (x) at these points.

$f'(-1.1) = 6(-1.1+1)(-1.1-2)$, which is positive i.e. $f'(x)>0$

$f'(-0.9) = 6(-0.9+1)(-0.9-2)$, which is negative i.e. $f'(x)<0$

Thus, at x = – 1, there is a local maximum.

Consider the point x =2.

Now, let us take x = 1.9 which is to the left of x = 2 and x = 2.1 which is to the right of x = 2 and find f '(x) at these points.

$$f'(1.9) = 6(1.9+1)(1.9-2)$$
$$= 6\times(\text{Positive number})\times(\text{negative number})$$
$$= \text{a negative number}$$

i.e. $\quad f'(1.9) < 0$

and $\quad f'(2.1)=6(2.1+1)(2.1-2)$, which is positive

i.e., $\quad f'(2.1)>0$

$f'(x) = 0$

$f'(x) < 0$ —— | —— $f'(x) > 0$

$x = 2$

$\because \quad f'(x) < 0$ as we approach 2 from the left

and $\quad f'(x) > 0$ as we approach 2 from the right.

$\therefore \quad x = 2$ is the point of local minimum

Thus f (x) has local maximum at x = –1, maximum value of f (x)=–2–3+12+8 =15

f (x) has local minimum at x =2, minimum value of f (x)=2(8)–3(4)–12(2)+8=–12

Sign of f ' (x)

Point x = – 1			Point x = 2	
Left of – 1	Right of – 1		Left of 2	Right of 2
positive	negative		negative	positive
local maximum			local minimum	

Example 29.33 Find local maximum and local minimum, if any, of the following function

$$f(x) = \frac{x}{1+x^2}$$

Solution :
$$f(x) = \frac{x}{1+x^2}$$

Then $$f'(x)=\frac{(1+x^2)1-(2x)x}{(1+x^2)^2}$$

$$=\frac{1-x^2}{(1+x^2)^2}$$

For finding points of local maximum or local minimum, equate f '(x) to 0.

i.e. $$\frac{1-x^2}{(1+x^2)^2}=0$$

$\Rightarrow$ $$1-x^2=0$$

or $(1+x)(1-x)=0$ or $x = 1,-1.$

Consider the value x = 1.

The sign of f '(x) for values of x slightly less than 1 and slightly greater than 1 changes from positive to negative. Therefore there is a local maximum at x =1, and the local maximum

value $=\dfrac{1}{1+(1)^2}=\dfrac{1}{1+1}=\dfrac{1}{2}$

Now consider x = – 1.

f '(x) changes sign from negative to positive as x passes through –1, therefore, f (x) has a local minimum at x = – 1

Thus, the local minimum value $=\dfrac{-1}{2}$

Example 29.34 Find the local maximum and local minimum, if any, for the function

$$f(x)=\sin x+\cos x, 0<x<\frac{\pi}{2}$$

Solution : We have $f(x)=\sin x+\cos x$

$$f'(x)=\cos x-\sin x$$

For local maxima/minima, $f'(x)=0$

$\therefore$ $$\cos x-\sin x=0$$

or, $\tan x=1$ or, $x=\dfrac{\pi}{4}$ in $0<x<\dfrac{\pi}{2}$

At $x=\dfrac{\pi}{4}$,

For $x<\dfrac{\pi}{4}$, $\cos x>\sin x$

$\therefore$ $$f'(x)=\cos x-\sin x>0$$

For $\quad x > \frac{\pi}{4}, \ \cos x - \sin x < 0$

$\therefore \quad f'(x) = \cos x - \sin x < 0$

$\therefore$ f '(x) changes sign from positive to negative in the neighbourhood of $\frac{\pi}{4}$.

$\therefore \ x = \frac{\pi}{4}$ is a point of local maxima.

Maximum value $= f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

$\therefore$ Point of local maxima is $\left(\frac{\pi}{4}, \sqrt{2}\right)$.

CHECK YOUR PROGRESS 29.9

Find all points of local maxima and local minima of the following functions. Also , find the maxima and minima at such points.

1. $x^2 - 8x + 12$
2. $x^3 - 6x^2 + 9x + 15$
3. $2x^3 - 21x^2 + 36x - 20$
4. $x^4 - 62x^2 + 120x + 9$
5. $(x-1)(x-2)^2$
6. $\frac{x-1}{x^2 + x + 2}$

29.10 USE OF SECOND DERIVATIVE FOR DETERMINATION OF MAXIMUM AND MINIMUM VALUES OF A FUNCTION

We now give below another method of finding local maximum or minimum of a function whose second derivative exists. Various steps used are :

(i) Let the given function be denoted by f (x).

(ii) Find f ' (x) and equate it to zero.

(iii) Solve f ' (x)=0, let one of its real roots be x = a.

(iv) Find its second derivative, f "(x). For every real value 'a' of x obtained in step (iii), evaluate f" (a). Then if

f "(a) <0 then x =a is a point of local maximum.

f "(a)>0 then x = a is a point of local minimum.

f "(a)=0 then we use the sign of f '(x) on the left of 'a' and on the right of 'a' to arrive at the result.

Example 29.35 Find the local minimum of the following function :

$$2x^3 - 21x^2 + 36x - 20$$

Solution : Let $\quad f(x) = 2x^3 - 21x^2 + 36x - 20$

Then $\quad f'(x) = 6x^2 - 42x + 36$

$$= 6\left(x^2 - 7x + 6\right)$$

$$= 6(x-1)(x-6)$$

For local maximum or min imum

$$f'(x) = 0$$

or $\quad 6(x-1)(x-6) = 0 \quad \Rightarrow x = 1, 6$

Now $\quad f''(x) = \dfrac{d}{dx} f'(x)$

$$= \frac{d}{dx}\left[6\left(x^2 - 7x + 6\right)\right]$$

$$= 12x - 42$$

$$= 6(2x - 7)$$

For $\quad x = 1,\ f''(1) = 6(2.1 - 7) = -30 < 0$

$x = 1$ is a point of local maximum.

and $f(1) = 2(1)^3 - 21(1)^2 + 36(1) - 20 = -3$ is a local maximum.

For $x = 6$,

$f''(6) = 6(2.6 - 7) = 30 > 0$

$\therefore \quad x = 6$ is a point of local minimum

and $f(6) = 2(6)^3 - 21(6)^2 + 36(6) - 20 = -128$ is a local minimum.

Example 29.36 Find local maxima and minima (if any) for the function

$$f(x) = \cos 4x; \quad 0 < x < \frac{\pi}{2}$$

Solution : $\quad f(x) = \cos 4x$

$\therefore \quad f'(x) = -4\sin 4x$

Now, $\quad f'(x) = 0 \quad \Rightarrow \quad -4\sin 4x = 0$

or, $\quad \sin 4x = 0 \quad$ or, $\quad 4x = 0, \pi, 2\pi$

or, $\quad x = 0, \frac{\pi}{4}, \frac{\pi}{2}$

$\therefore \quad x = \frac{\pi}{4} \quad \left[\because \; 0 < x < \frac{\pi}{2}\right]$

Now, $\quad f''(x) = -16\cos 4x$

at $\quad x = \frac{\pi}{4}, f''(x) = -16\cos\pi$

$$= -16(-1) = 16 > 0$$

$\therefore$ f (x) is minimum at $x = \frac{\pi}{4}$

Minimum value $\quad f\left(\frac{\pi}{4}\right) = \cos\pi = -1$

Example 29.37 (a) Find the maximum value of $2x^3 - 24x + 107$ in the interval [–3,–1].

(b) Find the minimum value of the above function in the interval [1,3].

Solution : Let $\quad f(x) = 2x^3 - 24x + 107$

$$f'(x) = 6x^2 - 24$$

For local maximum or minimum,

$$f'(x) = 0$$

i.e. $\quad 6x^2 - 24 = 0 \quad \Rightarrow \quad x = -2, 2$

Out of two points obtained on solving f '(x)=0, only–2 belong to the interval [–3,–1]. We shall, therefore, find maximum if any at x=–2 only.

Now $\quad f''(x) = 12x$

$\therefore \quad f''(-2) = 12(-2) = -24$

or $\quad f''(-2) < 0$

which implies the function f (x) has a maximum at x =–2.

$\therefore$ Required maximum value $\quad = 2(-2)^3 - 24(-2) + 107$

$$= 139$$

Thus the point of maximum belonging to the given interval [–3,–1] is –2 and, the maximum value of the function is 139.

(b) Now $\quad f''(x) = 12x$

$\therefore \qquad f''(2) = 24 > 0, \qquad [\because\ 2 \text{ lies in } [1, 3]]$

which implies, the function f (x) shall have a minimum at x =2.

$\therefore$ Required minimum $= 2(2)^3 - 24(2) + 107$

$= 75$

Example 29.38 Find the maximum and minimum value of the function

$$f(x) = \sin x(1+\cos x) \text{ in } (0, \pi).$$

Solution : We have, $f(x) = \sin x(1+\cos x)$

$$f'(x) = \cos x(1+\cos x) + \sin x(-\sin x)$$

$$= \cos x + \cos^2 x - \sin^2 x$$

$$= \cos x + \cos^2 x - (1-\cos^2 x) = 2\cos^2 x + \cos x - 1$$

For stationary points, $f'(x) = 0$

$$\therefore \qquad 2\cos^2 x + \cos x - 1 = 0$$

$$\therefore \qquad \cos x = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = -1, \frac{1}{2}$$

$$\therefore \qquad x = \pi, \frac{\pi}{3}$$

Now, $f(0) = 0$

$$f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3}\left(1+\cos\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$$

and $f(\pi) = 0$

$\therefore$ f (x) has maximum value $\dfrac{3\sqrt{3}}{4}$ at $x = \dfrac{\pi}{3}$

and miminum value 0 at $x = 0$ and $x = \pi$.

CHECK YOUR PROGRESS 29.10

Find local maximum and local minimum for each of the following functions using second order derivatives.

1. $2x^3 + 3x^2 - 36x + 10$
2. $-x^3 + 12x^2 - 5$

3. $(x-1)(x+2)^2$

4. $x^5 - 5x^4 + 5x^3 - 1$

5. $\sin x\,(1+\cos x),\, 0 < x < \frac{\pi}{2}$

6. $\sin x + \cos x,\, 0 < x < \frac{\pi}{2}$

7. $\sin 2x - x,\, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$

29.11 APPLICATIONS OF MAXIMA AND MINIMA TO PRACTICAL PROBLEMS

The application of derivative is a powerful tool for solving problems that call for minimising or maximising a function. In order to solve such problems, we follow the steps in the following order :

(i) Frame the function in terms of variables discussed in the data.

(ii) With the help of the given conditions, express the function in terms of a single variable.

(iii) Lastly, apply conditions of maxima or minima as discussed earlier.

Example 29.39 Find two positive real numbers whose sum is 70 and their product is maximum.

Solution : Let one number be x. As their sum is 70, the other number is 70–x. As the two numbers are positive, we have, $x > 0, 70 - x > 0$

$$70 - x > 0 \quad \Rightarrow \quad x < 70$$

$$\therefore \quad 0 < x < 70$$

Let their product be f (x)

Then $\quad f(x) = x(70 - x) = 70x - x^2$

We have to maximize the prouct f (x).

We, therefore, find f '(x) and put that equal to zero.

$$f'(x) = 70 - 2x$$

For maximum product, $f'(x) = 0$

or $\quad 70 - 2x = 0$

or $\quad x = 35$

Now $f''(x) = -2$ which is negative. Hence f (x) is maximum at x = 35

The other number is $70 - x = 35$

Hence the required numbers are 35, 35.

Example 29.40 Show that among rectangles of given area, the square has the least perimeter.

Solution : Let x, y be the length and breadth of the rectangle respectively.

$\therefore$ Its area = xy

Since its area is given, represent it by A, so that we have

$$A = xy$$

or $$y = \frac{A}{x} \qquad \text{... (i)}$$

Now, perimeter say P of the rectangle = 2 (x + y)

or $$P = 2\left(x + \frac{A}{x}\right)$$

$\therefore$ $$\frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right) \qquad \text{...(ii)}$$

For a minimum $P, \frac{dP}{dx} = 0.$

i.e. $$2\left(1 - \frac{A}{x^2}\right) = 0$$

or $$A = x^2 \quad \text{or} \quad \sqrt{A} = x$$

Now, $$\frac{d^2P}{dx^2} = \frac{4A}{x^3}, \text{ which is positive.}$$

Hence perimeter is minimum when $x = \sqrt{A}$

$\therefore$ $$y = \frac{A}{x}$$

$$= \frac{x^2}{x} = x \qquad (\because A = x^2)$$

Thus, the perimeter is minimum when rectangle is a square.

Example 29.41 An open box with a square base is to be made out of a given quantity of sheet of area a^2. Show that the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$.

Solution : Let x be the side of the square base of the box and y its height.

Total surface area of othe box $= x^2 + 4xy$

$\therefore$ $$x^2 + 4xy = a^2 \quad \text{or} \quad y = \frac{a^2 - x^2}{4x}$$

Volume of the box, $V = \text{base area} \times \text{height}$

$$= x^2 y = x^2\left(\frac{a^2 - x^2}{4x}\right)$$

or $$V = \frac{1}{4}\left(a^2 x - x^3\right) \qquad \text{...(i)}$$

$\therefore$ $$\frac{dV}{dx} = \frac{1}{4}\left(a^2 - 3x^2\right)$$

For maxima/minima $\frac{dV}{dx} = 0$

$\therefore$ $$\frac{1}{4}\left(a^2 - 3x^2\right) = 0$$

$$x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\text{Volume} = \frac{1}{4}\left(\frac{\left(a^3\right)}{\sqrt{3}} - \frac{a^3}{3\sqrt{3}}\right) = \frac{a^3}{6\sqrt{3}} \qquad \text{..(iii)}$$

Again $$\frac{d^2V}{dx^2} = \frac{d}{dx}\frac{1}{4}\left(a^2 - 3x^2\right) = -\frac{3}{2}x$$

x being the length of the side, is positive.

$\therefore$ $$\frac{d^2V}{dx^2} < 0$$

$\therefore$ The volume is maximum.

Hence maximum volume of the box $= \frac{a^3}{6\sqrt{3}}$.

Example 29.42 Show that of all rectangles inscribed in a given circle, the square has the maximum area.

Solution : Let ABCD be a rectangle inscribed in a circle of radius r. Then diameter AC= 2r

Let $AB = x$ and $BC = y$

Then $AB^2 + BC^2 = AC^2$ or $x^2 + y^2 = (2r)^2 = 4r^2$

Now area A of the rectangle = xy

$\therefore$ $$A = x\sqrt{4r^2 - x^2}$$

$\therefore$ $$\frac{dA}{dx} = \frac{x(-2x)}{2\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2} \cdot 1$$

$$= \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

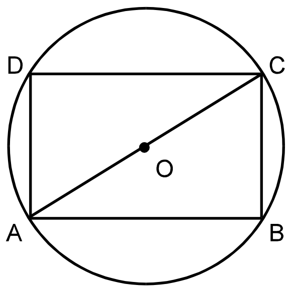


Fig. 29.10

For maxima/minima, $\frac{dA}{dx} = 0$

$$\frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0 \Rightarrow x = \sqrt{2}r$$

Now
$$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}\,(-4x) - \left(4r^2 - 2x^2\right)\frac{(-2x)}{2\sqrt{4r^2 - x^2}}}{\left(4r^2 - x^2\right)}$$

$$= \frac{-4x\left(4r^2 - x^2\right) + x\left(4r^2 - 2x^2\right)}{\left(4r^2 - x^2\right)^{\frac{3}{2}}}$$

$$= \frac{-4\sqrt{2}\left(2r^2\right) + 0}{\left(2r^2\right)^{\frac{3}{2}}} \qquad \text{... (Putting } x = \sqrt{2}r\text{)}$$

$$= \frac{-8\sqrt{2}r^3}{2\sqrt{2}r^3} = -4 < 0$$

Thus, A is maximum when $x = \sqrt{2}r$

Now, from (i), $y = \sqrt{4r^2 - 2r^2} = \sqrt{2}\ r$

x = y. Hence, rectangle ABCD is a square.

Example 29.43 Show that the height of a closed right circular cylinder of a given volume andleast surface is equal to its diameter.

Solution : Let V be the volume, r the radius and h the height of the cylinder.

Then, $V = \pi r^2 h$

or $h = \frac{V}{\pi r^2}$ ... (i)

Now surface area $S = 2\pi rh + 2\pi r^2$

$$= 2\pi r.\frac{V}{\pi r^2} + 2\pi r^2 = \frac{2V}{r} + 2\pi r^2$$

Now $$\frac{dS}{dr} = \frac{-2V}{r^2} + 4\pi r$$

For minimum surface area, $\frac{dS}{dr} = 0$

$\therefore$ $$\frac{-2V}{r^2} + 4\pi r = 0$$

or $$V = 2\pi r^3$$

From (i) and (ii), we get $$h = \frac{2\pi r^3}{\pi r^2} = 2r \quad ..(ii)$$

Again, $$\frac{d^2S}{dr^2} = \frac{4V}{r^3} + 4\pi = 8\pi + 4\pi \quad \text{... [Using (ii)]}$$

$$= 12\pi > 0$$

$\therefore$ S is least when h = 2r

Thus, height of the cylidner = diameter of the cylinder.

Example 29.44 Show that a closed right circular cylinder of given surface has maximum volume if its height equals the diameter of its base.

Solution : Let S and V denote the surface area and the volume of the closed right circular cylinder of height h and base radius r.

Then $$S = 2\pi rh + 2\pi r^2 \quad .....(i)$$
(Here surface is a constant quantity, being given)

$$V = \pi r^2 h$$

$\therefore$ $$V = \pi r^2 \left[\frac{S - 2\pi r^2}{2\pi r}\right]$$

$$= \frac{r}{2}\left[S - 2\pi r^2\right]$$

$$V = \frac{Sr}{2} - \pi r^3$$

$$\frac{dV}{dr} = \frac{S}{2} - \pi\left(3r^2\right)$$

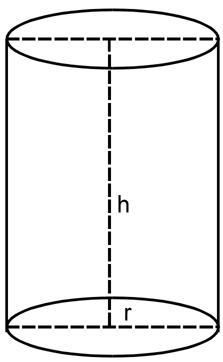


Fig. 29.11

For maximum or minimum, $\frac{dV}{dr} = 0$

i.e., $$\frac{S}{2} - \pi\left(3r^2\right) = 0$$

or $$S = 6\pi r^2$$

From (i), we have $$6\pi r^2 = 2\pi rh + 2\pi r^2$$

$$\Rightarrow \quad 4\pi r^2 = 2\pi rh$$

$$\Rightarrow \quad 2r = h \qquad ...(ii)$$

Also, $$\frac{d^2V}{dr^2} = \frac{d}{dr}\left[\frac{S}{2} - 3\pi r^2\right]$$

$$= -6\pi r, \qquad \because \frac{d}{dr}\left(\frac{S}{2}\right) = 0$$

= a negative quantity

Hence the volume of the right circular cylinder is maximum when its height is equal to twice its radius i.e. when h = 2r.

Example 29.45 A square metal sheet of side 48 cm. has four equal squares removed from the corners and the sides are then turned up so as to form an open box. Determine the size of the square cut so that volume of the box is maximum.

Solution : Let the side of each of the small squares cut be x cm, so that each side of the box to be made is (48–2x) cm. and height x cm.

Now $x > 0$, $48-2x > 0$, i.e. $x < 24$

$\therefore$ x lies between 0 and 24 or $0 < x < 24$

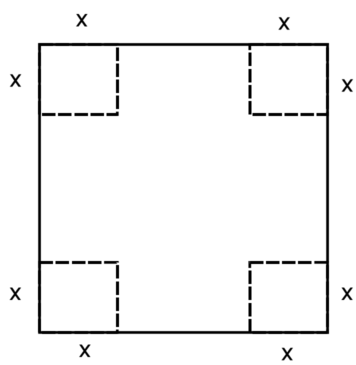


Fig. 29.12

Now, Volume V of the box

$$= (48-2x)(48-2x)x$$

i.e. $$V = (48-2x)^2 \cdot x$$

$$\therefore \quad \frac{dV}{dx} = (48-2x)^2 + 2(48-2x)(-2)x$$

$$= (48-2x)(48-6x)$$

Condition for maximum or minimum is $\frac{dV}{dx} = 0$

i.e., $(48-2x)(48-6x)=0$

We have either $x=24$, or $x=8$

$\because$ $0<x<24$

$\therefore$ Rejecting x = 24, we have, x = 8 cm.

Now, $$\frac{d^2V}{dx^2}=24x-384$$

$$\left(\frac{d^2V}{dx^2}\right)_{x=8}=192-384=-192<0,$$

Hence for x = 8, the volume is maximum.

Hence the square of side 8 cm. should be cut from each corner.

Example 29.46 The profit function P (x) of a firm, selling x items per day is given by

$$P(x)=(150-x)x-1625.$$

Find the number of items the firm should manufacture to get maximum profit. Find the maximum profit.

Solution : It is given that 'x' is the number of items produced and sold out by the firm every day. In order to maximize profit,

$$P'(x)=0 \text{ i.e. } \frac{dP}{dx}=0$$

or $$\frac{d}{dx}\left[(150-x)x-1625\right]=0$$

or $$150-2x=0$$

or $$x=75$$

Now, $\frac{d}{dx}P'(x)=P''(x)=-2=$ a negative quantity. Hence P (x) is maximum for x = 75.

Thus, the firm should manufacture only 75 items a day to make maximum profit.

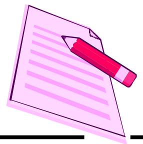
Now, Maximum Profit $=P(75)=(150-75)75-1625$

$= \text{Rs. } (75\times75-1625)$

$= \text{Rs. } (5625-1625)$

$= \text{Rs. } 4000$

MODULE - VIII
Calculus



Notes

Example 29.47 Find the volume of the largest cylinder that can be inscribed in a sphere of radius 'r' cm.

Solution : Let h be the height and R the radius of the base of the inscribed cylinder. Let V be the volume of the cylinder.

Then $V = \pi R^2 h$ ...(i)

From Δ OCB, we have

$$r^2 = \left(\frac{h}{2}\right)^2 + R^2 \qquad ...\left(\because OB^2 = OC^2 + BC^2\right)$$

$$\therefore \quad R^2 = r^2 - \frac{h^2}{4} \qquad ...(ii)$$

Now $$V = \pi\left(r^2 - \frac{h^2}{4}\right)h = \pi r^2 h - \pi\frac{h^3}{4}$$

$$\therefore \quad \frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4}$$

For maxima/minima, $\frac{dV}{dh} = 0$

$$\therefore \quad \pi r^2 - \frac{3\pi h^2}{4} = 0$$

$$\Rightarrow \quad h^2 = \frac{4r^2}{3} \quad \Rightarrow \quad h = \frac{2r}{\sqrt{3}}$$

Now $$\frac{d^2V}{dh^2} = -\frac{3\pi h}{2}$$

$$\therefore \quad \frac{d^2V}{dh^2}\left(\text{at } h = \frac{2r}{\sqrt{3}}\right) = -\frac{3\pi \times 2r}{2 \times \sqrt{3}}$$

$$= -\sqrt{3}\pi r < 0$$

$\therefore$ V is maximum at $h = \frac{2r}{\sqrt{3}}$

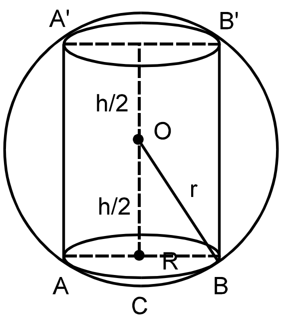


Fig. 29.13

Putting $h = \frac{2r}{\sqrt{3}}$ in (ii), we get

$$R^2 = r^2 - \frac{4r^2}{4 \times 3} = \frac{2r^2}{3}, \quad \therefore \quad R = \sqrt{\frac{2}{3}}\,r$$

Maximum volume of the cylinder $= \pi R^2 h$

$$= \pi \cdot \left(\frac{2}{3}r^2\right)\frac{2r}{\sqrt{3}} = \frac{4\pi r^3}{3\sqrt{3}}\,cm^3.$$

CHECK YOUR PROGRESS 29.11

1. Find two numbers whose sum is 15 and the square of one multiplied by the cube of the other is maximum.

2. Divide 15 into two parts such that the sum of their squares is minimum.

3. Show that among the rectangles of given perimeter, the square has the greatest area.

4. Prove that the perimeter of a right angled triangle of given hypotenuse is maximum when the triangle is isosceles.

5. A window is in the form of a rectangle surmounted by a semi-circle. If the perimeter be 30 m, find the dimensions so that the greatest possible amount of light may be admitted.

6. Find the radius of a closed right circular cylinder of volume 100 c.c. which has the minimum total surface area.

7. A right circular cylinder is to be made so that the sum of its radius and its height is 6 m. Find the maximum volume of the cylinder.

8. Show that the height of a right circular cylinder of greatest volume that can be inscribed in a right circular cone is one-third that of the cone.

9. A conical tent of the given capacity (volume) has to be constructed. Find the ratio of the height to the radius of the base so as to minimise the canvas requried for the tent.

10. A manufacturer needs a container that is right circular cylinder with a volume 16π cubic meters. Determine the dimensions of the container that uses the least amount of surface (sheet) material.

11. A movie theatre's management is considering reducing the price of tickets from Rs.55 in order to get more customers. After checking out various facts they decide that the average number of customers per day 'q' is given by the function where x is the amount of ticket price reduced. Find the ticket price othat result in maximum revenue.

$$q = 500+100\,x$$

where x is the amount of ticket price reduced. Find the ticket price that result is maximum revenue.