26

311en26

DIFFERENTIATION

The differential calculus was introduced sometime during 1665 or 1666, when Isaac Newton first concieved the process we now know as differentiation (a mathematical process and it yields a result called derivative). Among the discoveries of Newton and Leibnitz are rules for finding derivatives of sums, products and quotients of composite functions together with many other results. In this lesson we define derivative of a function, give its geometrical and physical interpretations, discuss various laws of derivatives and introduce notion of second order derivative of a function.

OBJECTIVES

After studying this lesson, you will be able to :

- define and interpret geometrically the derivative of a function $y = f(x)$ at $x = a$;
- prove that the derivative of a constant function $f(x) = c$, is zero;
- find the derivative of $f(x) = x^n, n \in Q$ from first principle and apply to find the derivatives of various functions;
- find the derivatives of the functions of the form $cf(x)$, $[f(x) \pm g(x)]$ and polynomial functions;
- state and apply the results concerning derivatives of the product and quotient of two functions;
- state and apply the chain rule for the derivative of a function;
- find the derivative of algebraic functions (including rational functions); and
- find second order derivative of a function.

EXPECTED BACKGROUND KNOWLEDGE

- Binomial Theorem
- Functions and their graphs
- Notion of limit of a function

26.1 DERIVATIVE OF A FUNCTION

Consider a function and a point say (5,25) on its graph. If x changes from 5 to 5.1, 5.01, 5.001..... etc., then correspondingly, y changes from 25 to 26.01, 25.1001, 25.010001,....A small change in x causes some small change in the value of y. We denote this change in the value of x by a symbol δx and the corresponding change caused in y by δy and call these respectively as an increment in x and increment in y, irrespective of sign of increment. The ratio $\frac{\delta x}{\delta y}$ of increment is termed as incrementary ratio. Here, observing the following table for $y = x^2$ at (5,25), we have for $\delta x = 0.1, 0.01, 0.001, 0.0001, ...$ $\delta y = 1.01, .1001, .010001, .00100001, ...$

x	5.1	5.01	5.001	5.0001
δx	.1	.01	.001	.0001
y	26.01	25.1001	25.010001	25.00100001
δy	1.01	.1001	.010001	.00100001
$\frac{\delta y}{\delta x}$	10.1	10.01	10.001	10.0001

We make the following observations from the above table :

(i) δy varies when δx varies.

(ii) $\delta y \to 0$ when $\delta x \to 0$.

(iii) The ratio $\frac{\delta y}{\delta x}$ tends to a number which is 10.

Hence, this example illustrates that $\delta y \to 0$ when $\delta x \to 0$ but $\frac{\delta y}{\delta x}$ tends to a finite number, not necessarily zero. The limit, $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ is equivalently represented by $\frac{dy}{dx}$. $\frac{dy}{dx}$ is called the derivative of y with respect to x and is read as differential coefficient of y with respect to x.

That is, $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} = 10$ in the above example and note that while δx and δy are small numbers (increments), the ratio $\frac{\delta y}{\delta x}$ of these small numbers approaches a definite value 10.

In general, let us consider a function

$$y = f(x) \qquad ...(i)$$

To find its derivative, consider δx to be a small change in the value of x, so $x + \delta x$ will be the new value of x where $f(x)$ is defined. There shall be a corresponding change in the value of y. Denoting this change by δy ; y + δy will be the resultant value of y, thus,

$$y + \delta y = f(x + \delta x) \qquad ...(ii)$$

Subtracting (i) from (ii), we have,

$$(y + \delta y) - y = f(x + \delta x) - f(x)$$

or $$\delta y = f(x + \delta x) - f(x) \qquad ...(iii)$$

To find the rate of change, we divide (iii) by δx

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} \qquad ...(iv)$$

Lastly, we consider the limit of the ratio $\frac{\delta y}{\delta x}$ as $\delta x \to 0$.

If $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \qquad ...(v)$$

is a finite quantity, then f(x) is called derivable and the limit is called derivative of f(x) with respect to (w.r.t.) x and is denoted by the symbol $f'(x)$ or by $\frac{d}{dx}$ of f(x)

i.e. $\frac{d}{dx} f(x)$ or $\frac{dy}{dx}$ (read as $\frac{d}{dx}$ of y).

Thus, $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x)$$

Remarks

(1) The limiting process indicated by equation (v) is a mathematical operation. This mathematical process is known as differentiation and it yields a result called a derivative.

(2) A function whose derivative exists at a point is said to be derivable at that point.

(3) It may be verified that if $f(x)$ is derivabale at a point $x = a$, then, it must be continuous at that point. However, the converse is not necessarily true.

(4) The symbols Δx and h are also used in place of δx i.e.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(5) If $y = f(x)$, then $\frac{dy}{dx}$ is also denoted by y_1 or y'.

26.2 VELOCITY AS LIMIT

Let a particle initially at rest at 0 moves along a strainght line OP., The distance s

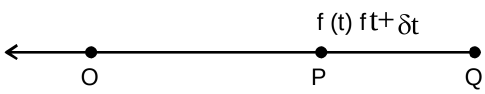

Fig. 26.1

covered by it in reaching P is a function of time t, We may write distance

$$OP = s = f(t) \qquad ...(i)$$

In the same way in reaching a point Q close to P covering PQ

i.e., δs is a fraction of time δt so that

$$OQ = OP + PQ$$

$$= s + \delta s$$

$$= f(t + \delta t) \qquad ...(ii)$$

The average velocity of the particle in the interval δt is given by

$$= \frac{\text{Change in distance}}{\text{Change in time}}$$

$$= \frac{(s+\delta s) - s}{(t+\delta t) - t}, \qquad \text{[From (i) and (ii)]}$$

$$= \frac{f\,(t+\delta t) - f(t)}{\delta t}$$

(average rate at which distance is travelled in the interval δt).

Now we make δt smaller to obtain average velocity in smaller interval near P. The limit of average velocity as $\delta t \to 0$ is the instantaneous velocity of the particle at time t (at the point P).

$\therefore$ Velocity at time $t = \lim\limits_{\delta t \to 0} \dfrac{f\,(t+\delta t) - f\,(t)}{\delta t}$

It is denoted by $\dfrac{ds}{dt}$.

Thus, if f (t) gives the distance of a moving particle at time t, then the derivative of 'f' at $t = t_0$ represents the instantaneous speed of the particle at the point P i.e. at time $t = t_0$.

This is also referred to as the physical interpretation of a derivative of a function at a point.

Note : The derivative $\dfrac{dy}{dx}$ represents instantaneous rate of change of y w.r.t. x.

Example 26.1 The distance 's' meters travelled in time t seconds by a car is given by the relation

$$s = 3t^2$$

Find the velocity of car at time t =4 seconds.

Solution : Here, $f\,(t) = s = 3t^2$

$$\therefore \qquad f\,(t+\delta t) = s + \delta s = 3(t+\delta t)^2$$

Velocity of car at any time $t = \lim\limits_{\delta t \to 0} \dfrac{f\,(t+\delta t) - f\,(t)}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{3(t+\delta t)^2 - 3t^2}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{3(t^2 + 2t.\delta t + \delta t^2) - 3t^2}{\delta t}$$

$$= \lim_{\delta t \to 0} (6t + 3\delta t)$$

$$= 6t$$

$\therefore$ Velocity of the car at t = 4 sec = (6 × 4) m/sec = 24 m/sec.

CHECK YOUR PROGRESS 26.1

1. Find the velocity of particles moving along a straight line for the given time-distance relations at the indicated values of time t :

(a) $s = 2 + 3t; t = \frac{1}{3}.$ (b) $s = 8t - 7; t = 4.$

(c) $s = t^2 + 3t; t = \frac{3}{2}.$ (d) $s = 7t^2 - 4t + 1; t = \frac{5}{2}.$

2. The distance s metres travelled in t seconds by a particle moving in a straight line is given by $s = t^4 - 18t^2$. Find its speed at t = 10 seconds.

3. A particle is moving along a horizontal line. Its distance s meters from a fixed point O at t seconds is given by $s = 10 - t^2 + t^3$. Determine its instantaneous speed at the end of 3 seconds.

26.3 GEOMETRICAL INTERPRETATION OF dy/dx

Let y = f (x) be a continuous function of x, draw its graph and denote it by APQB.

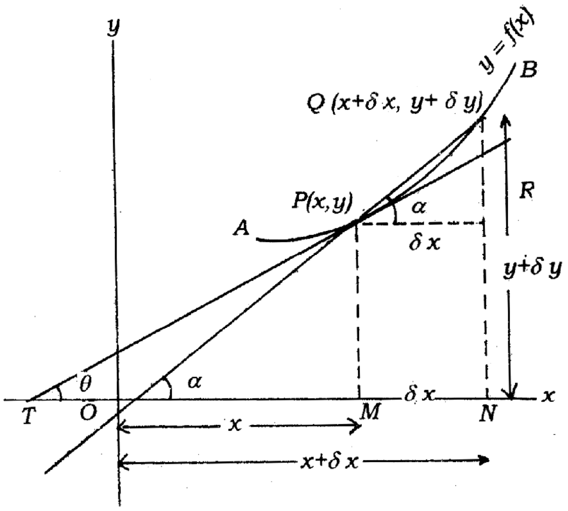

Fig. 26.2

Let P (x,y) be any point on the graph of y = f(x) or curve represented by y = f(x). Let $Q(x + \delta x, y + \delta y)$ be another point on the same curve in the neighbourhood of point P.

Draw PM and QN perpendiculars to x-axis and PR parallel to x-axis such that PR meets QN at R.Join QP and produce the secant line to any point S. Secant line QPS makes angle say α with the positive direction of x-axis. Draw PT tangent to the curve at the point P, making angle θ with the x-axis.

Now, In $\Delta QPR, \angle QPR = \alpha$

$$\tan\alpha = \frac{QR}{PR} = \frac{QN - RN}{MN} = \frac{QN - PM}{ON - OM} = \frac{(y + \delta y) - y}{(x + \delta x) - x} = \frac{\delta y}{\delta x} \quad \text{(i)}$$

Now, let the point Q move along the curve towards P so that Q approaches nearer and nearer the point P.

Thus, when $Q \to P$, $\delta x \to 0$, $\delta y \to 0$, $\alpha \to 0, (\tan\alpha \to \tan\theta)$ and consequently, the secant QPS tends to coincide with the tangent PT.

From (i). $$\tan\alpha = \frac{\delta y}{\delta x}$$

In the limiting case, $$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \alpha \to 0}} \tan\alpha = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

or $$\tan\theta = \frac{dy}{dx} \qquad \text{...(ii)}$$

Thus the derivative $\frac{dy}{dx}$ of the function $y = f(x)$ at any point P (x,y) on the curve represents the slope or gradient of the tangent at the point P.

This is called the geometrical interpretation of $\frac{dy}{dx}$.

It should be noted that $\frac{dy}{dx}$ has different values at different points of the curve.

Therefore, in order to find the gradient of the curve at a particular point, find $\frac{dy}{dx}$ from the equation of the curve $y = f(x)$ and substitute the coordinates of the point in $\frac{dy}{dx}$.

Corollary 1

If tangent to the curve at P is parallel to x-axis, then $\theta = 0^\circ$ or 180°, i.e., $\frac{dy}{dx} = \tan 0^\circ$ or $\tan 180^\circ$ i.e., $\frac{dy}{dx} = 0$.

That is tangent to the curve represented by $y = f(x)$ at P is parallel to x-axis.

Corollary 2

If tangent to the curve at P is perpendicular to x-axis, $\theta = 90^\circ$ or $\frac{dy}{dx} = \tan 90^\circ = \infty$.

That is, the tangent to the curve represented by $y = f(x)$ at P is parallel to y-axis.

26.4 DERIVATIVE OF CONSTANT FUNCTION

Statement : The derivative of a constant is zero.

Proof : Let $y = c$ be a constant function. Then $y = c$ can be written as

$$y = cx^0 \qquad [\because x^0 = 1] \qquad ..(i)$$

Let δx be a small increment in x. Corresponding to this increment, let δy be the increment in the value of y so that

$$y + \delta y = c(x + \delta x)^0 \qquad ..(ii)$$

Subtracting (i) from (ii),

$$(y + \delta y) - y = c(x + \delta x)^0 - cx^0, \qquad (\because x^0 = 1)$$

or $\qquad \delta y = c - c \qquad$ or $\qquad \delta y = 0$

Dividing by $\qquad \delta x$, $\quad \dfrac{\delta y}{\delta x} = \dfrac{0}{\delta x} \qquad$ or $\qquad \dfrac{\delta y}{\delta x} = 0$

Taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 0 \qquad \text{or} \qquad \frac{dy}{dx} = 0$$

or $$\frac{dc}{dx} = 0 \qquad [\, y = c \text{ from (i)}]$$

This proves that rate of change of constant quantity is zero. Therefore, derivative of a constant quantity is zero.

26.5 DERIVATIVE OF A FUNCTION FROM FIRST PRINCIPLE

Recalling the definition of derivative of a function at a point, we have the following working rule for finding the derivative of a function from first principle:

Step I. Write down the given function in the form of $y = f(x)$ ....(i)

Step II. Let dx be an increment in x, δy be the corresponding increment in y so that

$$y + \delta y = f(x + \delta x) \qquad ....(ii)$$

Step III. Subtracting (i) from (ii), we get

$$\delta y = f(x + \delta x) - f(x) \qquad ..(iii)$$

Step IV. Dividing the result obtained in step (iii) by δx, we get,

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Step V. Proceeding to limit as $\delta x \to 0$.

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Note : The method of finding derivative of function from first principle is also called **delta** or **ab-ininitio** method.

Next, we find derivatives of some standard and simple functions by first principle.

$$\text{Let } y = x^n \qquad \text{.....(i)}$$

For a small increment δx in x, let the corresponding increment in y be δy.

Then $$y + \delta y = (x + \delta x)^n . \qquad \text{...(ii)}$$

Subtracing (i) from (ii) we have,

$$(y + \delta y) - y = (x + \delta x)^n - x^n$$

$$\therefore \qquad \delta y = x^n \left(1 + \frac{\delta x}{x}\right)^n - x^n$$

$$= x^n \left[\left(1 + \frac{\delta x}{x}\right)^n - 1\right]$$

Since $\frac{\delta x}{x} < 1$, as δx is a small quantity compared to x, we can expand $\left(1 + \frac{\delta x}{x}\right)^n$ by Binomial theorem for any index.

Expanding $\left(1 + \frac{\delta x}{x}\right)^n$ by Binomial theorem, we have

$$\delta y = x^n \left[1 + n\left(\frac{\delta x}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{\delta x}{x}\right)^3 + \ldots - 1\right]$$

$$= x^n (\delta x)\left[\frac{n}{x} + \frac{n(n-1)}{2}\frac{\delta x}{x^2} + \frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3} + \ldots.\right]$$

Dividing by δx, we have

$$\frac{\delta y}{\delta x} = x^n \left[\frac{n}{x} + \frac{n(n-1)}{2!}\frac{\delta x}{x^2} + \frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3} + \ldots\right]$$

Proceeding to limit when $\delta x \to 0$, $(\delta x)^2$ and higher powers of δx will also tend to zero.

$$\therefore \qquad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^n \left[\frac{n}{x} + \frac{n(n-1)}{2!}\frac{\delta x}{x^2} + \frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3} + \ldots\right]$$

or $$\operatorname*{lt}_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} = x^n \left[\frac{n}{x} + 0 + 0 + \ldots\right]$$

or $$\frac{dy}{dx} = x^n \cdot \frac{n}{x} = nx^{n-1}$$

or $$\mathbf{\frac{d}{dx}(x^n) = nx^{n-1},} \qquad \left[\because \ y = x^n\right]$$

This is known as Newton's Power Formula or Power Rule

Note : We can apply the above formula to find derivative of functions like $x, x^2, x^3, ...$

i.e. when n = 1,2,3,...

e.g. $$\frac{d}{dx}x = \frac{d}{dx}x^1 = 1x^{1-1} = 1x^0 = 1.1 = 1$$

$$\frac{d}{dx}x^2 = 2x^{2-1} = 2x$$

$$\frac{d}{dx}\left(x^3\right) = 3x^{3-1} = 3x^2, \text{ and so on.}$$

Example 26.2 Find the derivative of each of the following :

(i) x^{10} (ii) x^{50} (iii) x^{91}

Solution :

(i)
$$\frac{d}{dx}\left(x^{10}\right) = 10x^{10-1} = 10x^9$$

(ii)
$$\frac{d}{dx}\left(x^{50}\right) = 50x^{50-1} = 50x^{49}$$

(iii)
$$\frac{d}{dx}\left(x^{91}\right) = 91x^{91-1} = 91x^{90}$$

We shall now find the derivatives of some simple functions from definition or first principles.

Example 26.3 Find the derivative of x^2 from the first principles.

Solution : Let $y = x^2$ (i)

For a small increment δx in x let the corresponding increment in y be δy.

$$y + \delta y = (x + \delta x)^2 \quad \text{(ii)}$$

Subtracting (i) from (ii), we have

$$(y + \delta y) - y = (x + \delta x)^2 - x^2$$

or $$\delta y = x^2 + 2x(\delta x) + (\delta x)^2 - x^2$$

or $$\delta y = 2x(\delta x) + (\delta x)^2$$

Divide by δx, we have

$$\frac{\delta y}{\delta x} = 2x + \delta x$$

Proceeding to limit when $\delta x \to 0$, we have

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}(2x + \delta x)$$

or $$\frac{dy}{dx} = 2x + \lim_{\delta x \to 0}(\delta x)$$

Notes

$$= 2x + 0$$
$$= 2x$$

or $\quad \dfrac{dy}{dx} = 2x \quad$ or $\quad \dfrac{d}{dx}\left(x^2\right) = 2x$

or $\quad \dfrac{dy}{dx} = \dfrac{-1}{x(x+0)} \quad$ or $\quad \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$

Example 26.4 Find the derivative of $\sqrt{x}$ by ab-initio method.

Solution : Let $\quad y = \sqrt{x} \qquad$ ..(i)

For a small increment δx in x, let δy be the corresponding increment in y.

$\therefore \qquad y + \delta y = \sqrt{x + \delta x} \qquad$ ...(ii)

Subtracting (i) from (ii), we have

$$(y+\delta y) - y = \sqrt{x+\partial x} - \sqrt{x} \qquad \text{...(iii)}$$

or $\qquad \delta y = \sqrt{x + \delta x} - \sqrt{x}$

Rationalising the numerator of the right hand side of (iii), we have

$$\delta y = \frac{\sqrt{x+\delta x} - \sqrt{x}}{\sqrt{x+\delta x} + \sqrt{x}}\left(\sqrt{x+\delta x} + \sqrt{x}\right)$$

$$= \frac{(x+\delta x) - x}{\sqrt{x+\delta x} + \sqrt{x}} \quad \text{or} \quad \delta y = \frac{\delta x}{\sqrt{x+\delta x} + \sqrt{x}}$$

Dividing by δx, we have

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x+\delta x} + \sqrt{x}}$$

Proceeding to limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left[\frac{1}{\sqrt{x+\delta} + \sqrt{x}}\right]$$

or $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x} + \sqrt{x}} \quad$ or $\quad \dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{1}{2\sqrt{x}}$

Example 26.5 If f (x) is a differentiable function and c is a constant, find the derivative of

$$\phi(x) = cf(x)$$

Solution : We have to find derivative of function

$$\phi(x) = cf(x) \qquad \text{...(i)}$$

For a small increment δx in x, let the values of the functions $\phi(x)$ be $\phi(x + \delta x)$ and that of f (x)be $f(x + \delta x)$

$\therefore \qquad \phi(x + \delta x) = cf(x + \delta x) \qquad$ ...(ii)

Subtracting (i) from (ii), we have

$$\phi(x+\delta x)-\phi(x)=c[f(x+\delta x)-f(x)]$$

Dividing by δx , we have

$$\frac{\phi(x+\delta x)-\phi(x)}{\delta x}=c\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

Proceeding to limit as $\delta x \to 0$, we have

$$\lim_{\delta x\to 0}\frac{\phi(x+\delta x)-\phi(x)}{\delta x}=\lim_{\delta x\to 0} c\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

or $$\phi'(x)=c\lim_{\delta x\to 0}\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

or $$\phi'(x)=cf'(x)$$

Thus, $$\frac{d}{dx}\left[cf(x)\right]=c\frac{df}{dx}$$

CHECK YOUR PROGRESS 26.2

1. Find the derivative of each of the following functions by delta method :

   (a) $10x$ (b) $2x+3$ (c) $3x^2$

   (d) x^2+5x (e) $7x^3$

2. Find the derivative of each of the following functions using ab-initio method:

   (a) $\frac{1}{x}, x \neq 0$ (b) $\frac{1}{ax}, x \neq 0$ (c) $x+\frac{1}{x}, x \neq 0$

   (d) $\frac{1}{ax+b}, x \neq \frac{-b}{a}$ (e) $\frac{ax+b}{cx+d}, x \neq \frac{-d}{c}$ (f) $\frac{x+2}{3x+5}, x \neq \frac{-5}{3}$

3. Find the derivative of each of the following functions from first principles :

   (a) $\frac{1}{\sqrt{x}}, x \neq 0$ (b) $\frac{1}{\sqrt{ax+b}}, x \neq \frac{-b}{a}$ (c) $\sqrt{x}+\frac{1}{\sqrt{x}}, x \neq 0$

   (d) $\frac{1+x}{1-x}, x \neq 1$

4. Find the derivative of each of the following functions by using delta method :

   (a) $f(x)=3\sqrt{x}$. Also find $f'(2)$. (b) $f(r)=\pi r^2$. Also find $f'(2)$.

   (c) $f(r)=\frac{4}{3}\pi r^3$. Also find $f'(3)$.

26.7 ALGEBRA OF DERIVATIVES

Many functions arise as combinations of other functions. The combination could be sum, difference, product or quotient of functions. We also come across situations where a given function can be expressed as a function of a function.

In order to make derivative as an effective tool in such cases, we need to establish rules for finding derivatives of sum, difference, product, quotient and function of a function. These, in turn, will enable one to find derivatives of polynomials and algebraic (including rational) functions.

26.7 DERIVATIVES OF SUM AND DIFFERENCE OF FUNCTIONS

If f (x) and g (x) are both derivable functions and h (x) = f (x) + g (x), then what is h' (x) ?

Here $\qquad h(x) = f(x) + g(x)$

Let δx be the increment in x and δy be the correponding increment in y.

$\therefore \qquad h(x+\delta x) = f(x+\delta x) + g(x+\delta x)$

Hence
$$h'(x) = \lim_{\delta x \to 0} \frac{[f(x+\delta x)+g(x+\delta x)]-[f(x)+g(x)]}{\delta x}$$
$$= \lim_{\delta x \to 0} \frac{[f(x+\delta x)-f(x)]+[g(x+\delta x)-g(x)]}{\delta x}$$
$$= \lim_{\delta x \to 0} \left[\frac{f(x+\delta x)-f(x)}{\delta x} + \frac{g(x+\delta x)-g(x)}{\delta x}\right]$$
$$= \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} + \lim_{\delta x \to 0} \frac{g(x+\delta x)-g(x)}{\delta x}$$

or $\qquad h'(x) = f'(x) + g'(x)$

Thus we see that the *derivative of sum of two functions is sum of their derivatives*.

This is called the **SUM RULE**.

e.g. $\qquad y = x^2 + x^3$

Then $\qquad y' = \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3)$

$$= 2x + 3x^2$$

Thus $\qquad y' = 2x + 3x^2$

This sum rule can easily give us the difference rule as well, because

if $\qquad h(x) = f(x) - g(x)$

then $\qquad h(x) = f(x) + [-g(x)]$

$$\therefore \qquad h'(x) = f'(x) + [-g'(x)]$$

$$= f'(x) - g'(x)$$

i.e. *the derivative of difference of two functions is the difference of their derivatives*.

This is called **DIFFERENCE RULE.**
Thus we have

Sum rule : $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

Difference rule : $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

Example 26.6 Find the derivative of each of the following functions :

(i) $y = 10t^2 + 20t^3$

(ii) $y = x^3 + \frac{1}{x^2} - \frac{1}{x}, \quad x \neq 0$

Solution :

(i) We have, $y = 10t^2 + 20t^3$

$$\therefore \qquad \frac{dy}{dt} = 10(2t) + 20(3t^2)$$

$$= 20t + 60t^2$$

(ii) $y = x^3 + \frac{1}{x^2} - \frac{1}{x} \qquad x \neq 0$

$$= x^3 + x^{-2} - x^{-1}$$

$$\therefore \qquad \frac{dy}{dx} = 3x^2 + (-2)x^{-3} - (-1)x^{-2} = 3x^2 - \frac{2}{x^3} + \frac{1}{x^2}$$

Example 26.7 Evaluate the derivative of

$$y = x^3 + 3x^2 + 4x + 5, \quad x = 1$$

Solution :

(ii) We have $y = x^3 + 3x^2 + 4x + 5$

$$\therefore \qquad \frac{dy}{dx} = \frac{d}{dx}\left[x^3 + 3x^2 + 4x + 5\right] = 3x^2 + 6x + 4$$

$$\therefore \qquad \left.\frac{dy}{dx}\right]_{x=1} = 3(1)^2 + 6(1) + 4 = 13$$

Notes

CHECK YOUR PROGRESS 26.3

1. Find y' when :
   (a) $y = 12$ (b) $y = 12x$ (c) $y = 12x + 12$

2. Find the derivatives of each of the following functions :

   (a) $f(x) = 20x^9 + 5x$ (b) $f(x) = -50x^4 - 20x^2 + 4$

   (c) $f(x) = 4x^3 - 9 - 6x^2$ (d) $f(x) = \frac{5}{9}x^9 + 3x$

   (e) $f(x) = x^3 - 3x^2 + 3x - \frac{2}{5}$ (f) $f(x) = \frac{x^8}{8} - \frac{x^6}{6} + \frac{x^4}{4} - 2$

   (g) $f(x) = \frac{2}{5}x^{\frac{2}{3}} - x^{\frac{-4}{5}} + \frac{3}{x^2}$ (h) $f(x) = \sqrt{x} - \frac{1}{\sqrt{x}}$

3. (a) If $f(x) = 16x + 2$, find $f'(0), f'(3), f'(8)$

   (b) If $f(x) = \frac{x^3}{3} - \frac{x^2}{2} + x - 16$, find $f'(-1), f'(0), f'(1)$

   (c) If $f(x) = \frac{x^4}{4} + \frac{3}{7}x^7 + 2x - 5$, find $f'(-2)$

   (d) Given that $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dr}$ and hence $\left.\frac{dV}{dr}\right]_{r=2}$

26.8 DERIVATIVE OF PRODUCT OF FUNCTIONS

You are all familiar with the four fundamental operations of Arithmetic : addition, subtraction, multiplication and division. Having dealt with the sum and the difference rules, we now consider the derivative of product of two functions.

Consider $\quad y = (x^2 + 1)^2$

This is same as $\quad y = (x^2 + 1)(x^2 + 1)$

So we need now to derive the way to find the derivative in such situation.

We write $\quad y = (x^2 + 1)(x^2 + 1)$

Let δx be the increment in x and δy the correrponding increment in y. Then

$$y + \delta y = [(x + \delta x)^2 + 1][(x + \delta x)^2 + 1)]$$

$$\Rightarrow \quad \delta y = [(x + \delta x)^2 + 1][(x + \delta x)^2 + 1)] - (x^2 + 1)(x^2 + 1)$$

$$= [(x + \delta x)^2 + 1][(x + \delta x)^2 - x^2)] + (x^2 + 1)[(x + \delta x)^2 + 1] - (x^2 + 1)(x^2 + 1)$$

$$=[(x+\delta x)^2+1][(x+\delta x)^2-x^2]+(x^2+1)[x+\delta x)^2+1-(x^2+1)]$$

$$=[(x+\delta x)^2+1][(x+\delta x)^2-x^2]+(x^2+1)[(x+\delta x)^2-x^2]$$

$$\therefore \qquad \frac{\delta y}{\delta x}=\left[(x+\delta x)^2+1\right]\cdot\left[\frac{(x+\delta x)^2-x^2}{\delta x}\right]+\left(x^2+1\right)\left[\frac{(x+\delta x)^2-x^2}{\delta x}\right]$$

$$=\left[(x+\delta x)^2+1\right]\cdot\left[\frac{2x\delta x+(\delta x)^2}{\delta x}\right]+\left(x^2+1\right)\left[\frac{2x\delta x+(\delta x)^2}{\delta x}\right]$$

$$=[(x+\delta x)^2+1](2x+\delta x)+(x^2+1)(2x+\delta x)$$

$$\therefore \qquad \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}[(x+\delta x)^2+1]\cdot[2x+\delta x]+\lim_{\delta x\to 0}(x^2+1)(2x+\delta x)$$

or $$\frac{dy}{dx}=\left(x^2+1\right)(2x)+\left(x^2+1\right)\cdot(2x)$$

$$=4x\left(x^2+1\right)$$

Let us analyse : $$\frac{dy}{dx}=\left(x^2+1\right)\underbrace{(2x)}_{\substack{\text{derivative}\\ \text{of } x^2+1}}+\left(x^2+1\right)\underbrace{(2x)}_{\substack{\text{derivative}\\ \text{of } x^2+1}}$$

Consider $\qquad y=x^3\cdot x^2$

Is $\qquad \dfrac{dy}{dx}=x^3\cdot(2x)+x^2\cdot\left(3x^2\right)$?

Let us check $\qquad x^3(2x)+x^2\left(3x^2\right)$

$$=2x^4+3x^4$$

$$=5x^4$$

We have $\qquad y=x^3\cdot x^2$

$$=x^5$$

$$\therefore \qquad \frac{dy}{dx}=5x^4$$

In general, if f (x) and g (x) are two functions of x then the derivative of their product is defined by

$$\frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+g(x)f'(x)$$

$$=\left[\text{Ist function}\right]\left[\frac{d}{dx}(\text{Second function})\right]+\left[\text{Second function}\right]\left[\frac{d}{dx}(\text{Ist function})\right]$$

which is read as derivative of product of two functions is equal to

= [Ist function] [Derivative of Second function] +

[Second function] [Derivative of Ist function]

This is called the **PRODUCT RULE.**

Example 26.8 Find $\frac{dy}{dx}$, if $y = 5x^6(7x^2 + 4x)$

Method I. Here y is a product of two functions.

$$\therefore \quad \frac{dy}{dx} = (5x^6)\cdot\frac{d}{dx}(7x^2 + 4x) + (7x^2 + 4x)\frac{d}{dx}(5x^6)$$

$$= (5x^6)(14x + 4) + (7x^2 + 4x)(30x^5)$$

$$= 70x^7 + 20x^6 + 210x^7 + 120x^6$$

$$= 280x^7 + 140x^6$$

Method II $\quad y = 5x^6(7x^2 + 4x)$

$$= 35x^8 + 20x^7$$

$$\therefore \quad \frac{dy}{dx} = 35 \times 8x^7 + 20 \times 7x^6 = 280x^7 + 140x^6$$

which is the same as in Method I.

This rule can be extended to find the derivative of two or more than two functions.

Remark : If f (x), g (x) and h(x) are three given functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f(x)g(x)\frac{d}{dx}h(x) + g(x)h(x)\frac{d}{dx}f(x) + h(x)f(x)\frac{d}{dx}g(x)$$

Example 26.9 Find the derivative of $[f(x)g(x)h(x)]$ if

$$f(x) = x,\ g(x) = (x-3),\ \text{and}\quad h(x) = x^2 + x$$

Solution : Let $y = x(x-3)(x^2 + x)$

To find the derivative of y, we can combine any two functions, given on the R.H.S. and apply the product rule or use result mentioned in the above remark.

In other words, we can write

$$y = [x(x-3)](x^2 + x)$$

Let $\quad u(x) = f(x)g(x) = x(x-3) = x^2 - 3x$

Also $\quad h(x) = x^2 + x$

$\therefore \quad y = u(x) \times h(x)$

Hence $\quad \frac{dy}{dx} = x(x-3)\frac{d}{dx}(x^2 + x) + (x^2 + x)\frac{d}{dx}(x^2 - 3x)$

$$= x(x-3)(2x+1)+\left(x^2+x\right)(2x-3)$$

$$= x(x-3)(2x+1)+\left(x^2+x\right)(x-3)+x\left(x^2+x\right)$$

$$= [f(x)g(x)]\cdot h'(x)+[g(x)h(x)]f'(x)+[h(x)f(x)].g'(x)$$

Hence $\quad \frac{d}{dx}[f(x)g(x)h(x)] = [f(x)g(x)]\cdot\frac{d}{dx}[h(x)]$

$$+[g(x)h(x)]\frac{d}{dx}[f(x)]+h(x)f(x)\frac{d}{dx}[g(x)]$$

Alternatively, we can directly find the derivative of product of the given three functions.

$$\frac{dy}{dx} = [x(x-3)]\frac{d}{dx}(x^2+x)+[(x-3)(x^2+x)]\frac{d}{dx}(x)+[(x^2+x)\cdot x]\frac{d}{dx}(x-3)$$

$$= x(x-3)(2x+1)+(x-3)(x^2+x)\cdot 1+(x^2+x)\cdot x\cdot 1$$

$$= 4x^3-6x^2-6x$$

CHECK YOUR PROGRESS 26.4

1. Find the derivative of each of the following functions by product rule :

(a) $f(x) = (3x+1)(2x-7)$ (b) $f(x) = (x+1)(-3x-2)$

(c) $f(x) = (x+1)(-2x-9)$ (d) $y = (x-1)(x-2)$

(e) $y = x^2(2x^2+3x+8)$ (f) $y = (2x+3)(5x^2-7x+1)$

(g) $u(x) = (x^2-4x+5)(x^3-2)$

2. Find the derivative of each of the functions given below :

(a) $f(r) = r(1-r)(\pi r^2+r)$ (b) $f(x) = (x-1)(x-2)(x-3)$

(c) $f(x) = (x^2+2)(x^3-3x^2+4)(x^4-1)$

(d) $f(x) = (3x^2+7)(5x-1)\left(3x^2+9x+8\right)$

26.9 QUOTIENT RULE

You have learnt sum Rule, Difference Rule and Product Rule to find derivative of a function expressed respectively as either the sum or difference or product of two functions. Let us now take a step further and learn the "Quotient Rule for finding derivative of a function which is the quotient of two functions.

Let $\quad g(x) = \dfrac{1}{r(x)}, \quad [r(x)\neq 0]$

Let us find the derivative of g (x) by first principles

$$g(x) = \frac{1}{r(x)}$$

$$\therefore \qquad g'(x) = \lim_{\delta x \to 0}\left[\frac{\frac{1}{r(x+\delta x)} - \frac{1}{r(x)}}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{r(x) - r(x+\delta x)}{\delta(x) r(x) r(x+\delta x)}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{r(x) - r(x+\delta x)}{\delta x}\right] \lim_{\delta x \to 0}\frac{1}{r(x).r(x+\delta x)}$$

$$= -r'(x)\cdot\frac{1}{[r(x)]^2} = -\frac{r'(x)}{[r(x)]^2}$$

Consider any two functions f (x) and g (x) such that $\phi(x) = \frac{f(x)}{g(x)}$, $\qquad g(x) \neq 0$

We can write $\phi(x) = f(x)\cdot\frac{1}{g(x)}$

$$\therefore \qquad \phi(x) = f'(x)\cdot\frac{1}{g(x)} + f(x)\frac{d}{dx}\left[\frac{1}{g(x)}\right]$$

$$= \frac{f'(x)}{g(x)} + f(x)\left[\frac{-g'(x)}{[g(x)]^2}\right]$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{(\text{Denominator})(\text{Derivative of Numerator}) - (\text{Numerator})(\text{Derivative of Denominator})}{(\text{Denominator})^2}$$

Hence $\qquad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

This is called the quotient Rule.

Example 26.10 Find $f'(x)$ if $f(x) = \frac{4x+3}{2x-1}$, $\quad x \neq \frac{1}{2}$

Solution :

$$f'(x) = \frac{(2x-1)\frac{d}{dx}(4x+3) - (4x+3)\frac{d}{dx}(2x-1)}{(2x-1)^2}$$

$$= \frac{(2x-1).4-(4x+3).2}{(2x-1)^2}$$

$$= \frac{-10}{(2x-1)^2}$$

Let us consider the following example:

Let $\qquad f(x) = \frac{1}{2x-1}, \qquad x \neq \frac{1}{2}$

$$\frac{d}{dx}\left[\frac{1}{2x-1}\right] = \frac{(2x-1)\frac{d}{dx}(1) - 1\frac{d}{dx}(2x-1)}{(2x-1)^2}$$

$$= \frac{(2x-1)\times 0 - 2}{(2x-1)^2} \qquad \left[\because \frac{d}{dx}(1) = 0\right]$$

i.e. $\qquad \frac{d}{dx}\left[\frac{1}{2x-1}\right] = -\frac{2}{(2x-1)^2}$

CHECK YOUR PROGRESS 26.5

1. Find the derivative of each of the following :

(a) $y = \frac{2}{5x-7}, \ x \neq \frac{7}{5}$ (b) $y = \frac{3x-2}{x^2+x-1}$ (c) $y = \frac{x^2-1}{x^2+1}$

(d) $f(x) = \frac{x^4}{x^2-3}$ (e) $f(x) = \frac{x^5-2x}{x^7}$ (f) $f(x) = \frac{x}{x^2+x+1}$

(g) $f(x) = \frac{\sqrt{x}}{x^3+4}$

2. Find $f'(x)$ if

(a) $f(x) = \frac{x(x^2+3)}{x-2}, \qquad [x \neq 2]$

(b) $f(x) = \frac{(x-1)(x-2)}{(x-3)(x-4)}, \qquad [x \neq 3, \quad x \neq 4]$

26.10 CHAIN RULE

Earlier, we have come across functions of the type $\sqrt{x^4+8x^2+1}$. This function can not be expressed as a sum, difference, product or a quotient of two functions. Therefore, the techniques developed so far do not help us find the derivative of such a function. Thus, we need to develop a rule to find the derivative of such a function.

Let us write : $y = \sqrt{x^4 + 8x^2 + 1}$ or $y = \sqrt{t}$ where $t = x^4 + 8x^2 + 1$

That is, y is a function of t and t is a function of x. Thus y is a function of a function. We proceed to find the derivative of a function of a function.

Let δt be the increment in t and δy, the corresponding increment in y.

Then $\delta y \to 0$ as $\delta t \to 0$

$$\therefore \qquad \frac{dy}{dt} = \lim_{\delta t \to 0} \frac{\delta y}{\delta t} \qquad \text{(i)}$$

Similarly t is a function of x.

$\therefore$ $\delta t \to 0$ as $\delta x \to 0$

$$\therefore \qquad \frac{dt}{dx} = \lim_{\delta x \to 0} \frac{\delta t}{\delta x} \qquad \text{(ii)}$$

Here y is a function of t and t is a function of x. Therefore $\delta y \to 0$ as $\delta x \to 0$

From (i) and (ii), we get

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \left[\lim_{\delta t \to 0} \frac{\delta y}{\delta t}\right]\left[\lim_{\delta x \to 0} \frac{\delta t}{\delta x}\right]$$

$$= \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Thus $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

This is called the **Chain Rule.**

Example 26.11 If $y = \sqrt{x^4 + 8x^2 + 1}$, find $\frac{dy}{dx}$

Solution : We are given that

$$y = \sqrt{x^4 + 8x^2 + 1}$$

which we may write as

$$y = \sqrt{t}, \text{ where } t = x^4 + 8x^2 + 1 \qquad \text{(i)}$$

$$\therefore \qquad \frac{dy}{dt} = \frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx} = 4x^3 + 16x$$

Here $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{2\sqrt{t}} \cdot (4x^3 + 16x)$$

$$= \frac{4x^3 + 16x}{2\sqrt{x^4 + 8x^2 + 1}} = \frac{2x^3 + 8x}{\sqrt{x^4 + 8x^2 + 1}} \qquad \text{(Using (i))}$$

Example 26.12 Find the derivative of the function $y = \dfrac{5}{(x^2-3)^7}$

Solution :
$$\frac{dy}{dx} = \frac{d}{dx}\left\{5(x^2-3)^{-7}\right\}$$

$$= 5[(-7)(x^2-3)^{-8}]\cdot\frac{d}{dx}(x^2-3) \qquad \text{(Using chain Rule)}$$

$$= -35(x^2-3)^{-8}\cdot(2x)$$

$$= \frac{-70x}{(x^2-3)^8}$$

Example 26.13 Find $\dfrac{dy}{dx}$ where $y = \dfrac{1}{4}v^4$ and $v = \dfrac{2}{3}x^3+5$

Solution : We have $y = \dfrac{1}{4}v^4$ and $v = \dfrac{2}{3}x^3+5$

$$\frac{dy}{dv} = \frac{1}{4}(4v^3) = v^3 = \left(\frac{2}{3}x^3+5\right)^3 \qquad \text{...(i)}$$

and
$$\frac{dv}{dx} = \frac{2}{3}(3x^2) = 2x^2 \qquad \text{..(ii)}$$

Thus
$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx}$$

$$= \left(\frac{2}{3}x^3+5\right)^3(2x^2) \qquad \text{[Using (i) and (ii)]}$$

Remark

We have seen in the previous examples that by using various rules of derivatives we can find derivatives of algebraic functions.

CHECK YOUR PROGRESS 26.6

1. Find the derivative of each of the following functions :

(a) $f(x) = (5x-3)^7$ (b) $f(x) = (3x^2-15)^{35}$

(c) $f(x) = (1-x^2)^{17}$ (d) $f(x) = \dfrac{(3-x)^5}{7}$

(e) $y = \dfrac{1}{x^2+3x+1}$ (f) $y = \sqrt[3]{(x^2+1)^5}$

(g) $y = \dfrac{1}{\sqrt{7-3x^2}}$ (h) $y = \left[\dfrac{1}{6}x^6 + \dfrac{1}{2}x^4 + \dfrac{1}{16}\right]^5$

(i) $y = (2x^2 + 5x - 3)^{-4}$ (j) $y = x + \sqrt{x^2 + 8}$

2. Find $\dfrac{dy}{dx}$ if

(a) $y = \dfrac{3-v}{2+v}, v = \dfrac{4x}{1-x^2}$ (b) $y = at^2, t = \dfrac{x}{2a}$

Second Order Derivative : Given y is a function of x, say f (x). If the derivative $\dfrac{dy}{dx}$ is a derivable function of x, then the derivative of $\dfrac{dy}{dx}$ is known as the second derivative of $y = f(x)$ with respect to x and is denoted by $\dfrac{d^2y}{dx^2}$. Other symbols used for the second derivative of y are D^2, f'', y'', y_2 etc.

Remark

Thus the value of f '' at x is given by

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(h)}{h}$$

The derivatives of third, fourth, ...orders can be similarly defined.

Thus the second derivative, or second order derivative of y with respect to x is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example 26.14 Find the second order derivative of

(i) x^2 (ii) $x^3 + 1$ (iii) $(x^2+1)(x-1)$ (iv) $\dfrac{x+1}{x-1}$

Solution : (i) Let $y = x^2$, then $\dfrac{dy}{dx} = 2x$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x) = 2 \cdot \frac{d(x)}{dx}$$
$$= 2.1 = 2$$

$\therefore$ $$\frac{d^2y}{dx^2} = 2$$

(ii) Let $y = x^3 + 1$, then

$$\frac{dy}{dx} = 3x^2 \text{ (by sum rule and derivative of a constant is zero)}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2) = 3.2x = 6x$$

$\therefore$ $$\frac{d^2y}{dx^2} = 6x$$

(iii) Let $y = (x^2 + 1)(x - 1)$, then

$$\frac{dy}{dx} = (x^2 + 1)\frac{d}{dx}(x - 1) + (x - 1),\frac{d}{dx}(x^2 + 1)$$

$$= (x^2 + 1)\cdot 1 + (x - 1)\cdot 2x \quad \text{or} \quad \frac{dy}{dx} = x^2 + 1 + 2x^2 - 2x = 3x^2 - 2x + 1$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3x^2 - 2x + 1\right) = 6x - 2$$

$\therefore$ $$\frac{d^2y}{dx^2} = 6x - 2$$

(iv) Let $y = \dfrac{x + 1}{x - 1}$, then

$$\frac{dy}{dx} = \frac{(x - 1)\cdot 1 - (x + 1).1}{(x - 1)^2} = \frac{-2}{(x - 1)^2}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{-2}{(x - 1)^2}\right] = -2.-2.\frac{1}{(x - 1)^3} = \frac{4}{(x - 1)^3}$$

$\therefore$ $$\frac{d^2y}{dx^2} = \frac{4}{(x - 1)^3}$$

CHECK YOUR PROGRESS 26.7

Find the derivatives of second order for each of the following functions :

(a) x^3 (b) $x^4 + 3x^3 + 9x^2 + 10x + 1$

(c) $\dfrac{x^2 + 1}{x + 1}$ (d) $\sqrt{x^2 + 1}$

LET US SUM UP

- The derivative of a function f (x) with respect to x is defined as

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}, \ \delta x > 0$$

- The derivative of a constant is zero i.e., $\frac{dc}{dx} = 0$, where c a is constant.
- Newton's Power Formula

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

- Geometrically, the derivative $\frac{dy}{dx}$ of the function $y = f(x)$ at point $P(x, y)$ is the slope or gradient of the tangent on the curve represented by $y = f(x)$ at the point P.
- The derivative of y with respect to x is the instantaneous rate of change of y with respect to x.
- If f (x) is a derivable function and c is a constant, then

$\frac{d}{dx}\left[cf(x)\right] = cf'(x)$, where $f'(x)$ denotes the derivative of $f(x)$.

- 'Sum or difference rule' of functions :

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

Derivative of the sum or difference of two functions is equal to the sum or diference of their derivatives respectively.

- **Product rule :**

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

$$= (\text{Ist function})\left(\frac{d}{dx}\text{IInd function}\right) + (\text{IInd function})\left(\frac{d}{dx}\text{Ist function}\right)$$

- **Quotient rule :** If $\phi(x) = \frac{f(x)}{g(x)}, \ g(x) \neq 0$, then

$$\phi'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

$$= \frac{(\text{Denominator})\left(\frac{d}{dx}(\text{Numerator})\right) - \text{Numerator}\left(\frac{d}{dx}(\text{Denominator})\right)}{(\text{Denominator})^2}$$

- **Chain Rule :** $\frac{d}{dx}\left[f\{g(x)\}\right] = f'[g(x)] \cdot \frac{d}{dx}[g(x)]$

  $=$derivative of $f(x)$ w.r.t $g(x) \times$ derivative of $g(x)$ w.r.t.x

- The derivative of second order of y w.r.t. to x is $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=MKWBx78L7Qg

http://www.youtube.com/watch?v=IiBC4ngwH6E

http://www.youtube.com/watch?v=1015d63VKh4

http://www.youtube.com/watch?v=Bkkk0RLSEy8

http://www.youtube.com/watch?v=ho87DN9wO70

http://www.youtube.com/watch?v=UXQGzgPf1LE

http://www.youtube.com/watch?v=4bZyfvKazzQ

http://www.bbc.co.uk/education/asguru/maths/12methods/03differentiation/index.shtml

TERMINAL EXERCISE

1. The distance s meters travelled in time t seconds by a car is given by the relation $s = t^2$. Caclulate.

   (a) the rate of change of distance with respect to time t.

   (b) the speed of car at time t = 3 seconds.

2. Given $f(t) = 3 - 4t^2$. Use delta method to find $f'(t), f'\left(\frac{1}{3}\right)$.

3. Find the derivative of $f(x) = x^4$ from the first principles. Hence find

   $$f'(0), f'\left(-\frac{1}{2}\right)$$

4. Find the derivative of the function $\sqrt{2x+1}$ from the first principles.

5. Find the derivatives of each of the following functions by the first principles :

   (a) ax + b, where a and b are constants (b) $2x^2 + 5$

   (c) $x^3 + 3x^2 + 5$ (d) $(x-1)^2$

6. Find the derivative of each of the following functions :

Notes

(a) $f(x) = px^4 + qx^2 + 7x - 11$ (b) $f(x) = x^3 - 3x^2 + 5x - 8$

(c) $f(x) = x + \frac{1}{x}$ (d) $f(x) = \frac{x^2 - a}{a - 2}, a \neq 2$

7. Find the derivative of each of the functions given below by two ways, first by product rule, and then by expanding the product. Verify that the two answers are the same.

(a) $y = \sqrt{x}\left(1 + \frac{1}{\sqrt{x}}\right)$ (b) $y = x^{\frac{3}{2}}\left(2 + 5x + \frac{1}{x}\right)$

8. Find the derivative of the following functions :

(a) $f(x) = \frac{x}{x^2 - 1}$ (b) $f(x) = \frac{3}{(x-1)^2} + \frac{10}{x^3}$

(c) $f(x) = \frac{1}{(1 + x^4)}$ (d) $f(x) = \frac{(x+1)(x-2)}{\sqrt{x}}$

(e) $f(x) = \frac{3x^2 + 4x - 5}{x}$ (f) $f(x) = \frac{x - 4}{2\sqrt{x}}$

(g) $f(x) = \frac{(x^3 + 1)(x - 2)}{x^2}$

9. Use chain rule, to find the derivative of each of the functions given below :

(a) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ (b) $\sqrt{\frac{1+x}{1-x}}$ (c) $\sqrt[3]{x^2(x^2 + 3)}$

10. Find the derivatives of second order for each of the following :

(a) $\sqrt{x + 1}$ (b) $x \cdot \sqrt{x - 1}$

ANSWERS

CHECK YOUR PROGRESS 26.1

1. (a) 3 (b) 8 (c) 6 (d) 31

2. 3640 m/s 3. 21 m/s

CHECK YOUR PROGRESS 26.2

1. (a) 10 (b) 2 (c) 6x (d) 2x+5 (e) $21x^2$

2. (a) $-\frac{1}{x^2}$ (b) $-\frac{1}{ax^2}$ (c) $1-\frac{1}{x^2}$ (d) $\frac{-a}{(ax+b)^2}$ (e) $\frac{ad-bc}{(cx+d)^2}$ (f) $-\frac{1}{(3x+5)^2}$

3. (a) $-\frac{1}{2x\sqrt{x}}$ (b) $\frac{-a}{2(ax+b)(\sqrt{ax+b})}$ (c) $\frac{1}{2\sqrt{x}}\left(1-\frac{1}{x}\right)$ (d) $\frac{2}{(1-x)^2}$

4. (a) $\frac{3}{2\sqrt{x}};\frac{3}{2\sqrt{2}}$ (b) $2\pi r$; 4π (c) $2\pi r^2$; 36π

CHECK YOUR PROGRESS 26.3

1. (a) 0 (b) 12 (c) 12

2. (a) $180x^8+5$ (b) $-200x^3-40x$ (c) $12x^2-12x$ (d) $5x^8+3$ (e) $3x^2-6x+3$

   (f) $x^7-x^5+x^3$ (g) $\frac{4}{15}x^{\frac{-1}{3}}+\frac{4}{5}x^{\frac{-9}{5}}-6x^{-3}$ (h) $\frac{1}{2\sqrt{x}}+\frac{1}{2x^{\frac{3}{2}}}$

3. (a) 16, 16, 16 (b) 3,1 ,1 (c) 186 (d) $4\pi r^2, 16\pi$

CHECK YOUR PROGRESS 26.4

1. (a) $12x-19$ (b) $-6x-5$ (c) $4x-11$ (d) $2x-3$

   (e) $8x^3+9x^2+16x$ (f) $30x^2+2x-19$ (g) $5x^4-16x^3+15x^2-4x+8$

2. (a) $-4\pi r^3+3(\pi-1)r^2+2r$ (b) $3x^2-12x+11$

   (c) $9x^8-28x^7+14x^6-12x^5-5x^4+44x^3-6x^2+4x$

   (d) $(5x-1)(3x^2+9x+8).6x+5(3x^2+7)(3x^2+9x+8)+(3x^2+7)(5x-1)(6x+9)$

CHECK YOUR PROGRESS 26.5

1. (a) $\frac{-10}{(5x-7)^2}$ (b) $\frac{-3x^2+4x-1}{(x^2+x+1)^2}$ (c) $\frac{4x}{(x^2+1)^2}$

   (d) $\frac{2x^5-12x^3}{(x^2-3)^2}$ (e) $\frac{-2x^4+12}{x^7}$ (f) $\frac{1-x^2}{(x^2+x+1)^2}$ (g) $\frac{4-5x^3}{2\sqrt{x}(x^3+4)^2}$

2. (a) $\dfrac{2x^3-6x^2-6}{(x-2)^2}$ (b) $\dfrac{-4x^2+20x-22}{(x-3)^2(x-4)^2}$

CHECK YOUR PROGRESS 26.6

1. (a) $35(5x-6)^6$ (b) $210x(3x^2-15)^{34}$ (c) $-34x(1-x^2)^{16}$

   (d) $\dfrac{-5}{7}(3-x)^4$ (e) $-(2x+3)(x^2+3x+1)^{-2}$ (f) $\dfrac{10x}{3}\left(x^2+1\right)^{\frac{2}{3}}$

   (g) $3x(7-3x^2)^{-3/2}$ (h) $5(x^5+2x^3)\left(\dfrac{x^6}{6}+\dfrac{x^4}{2}+\dfrac{1}{16}\right)^4$

   (i) $-4(4x+5)(2x^2+5x-3)^{-5}$ (j) $1+\dfrac{x}{\sqrt{x^2+8}}$

2. (a) $\dfrac{-5(1+x^2)}{(1+2x-x^2)^2}$ (b) $\dfrac{x}{2a}$

CHECK YOUR PROGRESS 26.7

1. (a) $6x$ (b) $12x^2+18x+18$ (c) $\dfrac{4}{(x+1)^3}$ (d) $\dfrac{1}{(1+x^2)^{3/2}}$

TERMINAL EXERCISE

1. (a) 2 t (b) 6 seconds

2 $-8t, -\dfrac{8}{3}$ 3. $0, \dfrac{-1}{2}$ 4. $\dfrac{1}{\sqrt{2x+1}}$

5. (a) a (b) $4x$ (c) $3x^2+6x$ (d) $2(x-1)$

6. (a) $4px^3+2qx+7$ (b) $3x^2-6x+5$ (c) $1-\dfrac{1}{x^2}$ (d) $\dfrac{2x}{a-2}$

7. (a) $\dfrac{1}{2\sqrt{x}}$ (b) $3\sqrt{x}+\dfrac{25}{2}x\sqrt{x}+\dfrac{1}{2\sqrt{x}}$

8. (a) $\dfrac{-(x^2+1)}{(x^2-1)^2}$ (b) $\dfrac{-6}{(x-1)^3}-\dfrac{30}{x^4}$ (c) $\dfrac{-4x^3}{(1+x^4)^2}$ (d) $\dfrac{3}{2}\sqrt{x}-\dfrac{1}{2\sqrt{x}}+\dfrac{1}{x^{3/2}}$

   (e) $3+\dfrac{5}{x^2}$ (f) $\dfrac{1}{4\sqrt{x}}+\dfrac{1}{x\sqrt{x}}$ (g) $3x^2-2-\dfrac{1}{x^2}+\dfrac{4}{x^3}$

9. (a) $1-\dfrac{1}{x^2}$ (b) $\dfrac{1}{\sqrt{1+x}\cdot(1-x)^{\frac{3}{2}}}$ (c) $\dfrac{4x^3+6x}{3(x^4+3x^2)^{\frac{2}{3}}}$

10. (a) $-\dfrac{1}{4(x+1)^{\frac{3}{2}}}$ (b) $\dfrac{2+x-x^2}{4(x-1)^{\frac{1}{2}}}$