211en03

3

ALGEBRAIC EXPRESSIONS AND POLYNOMIALS

So far, you had been using arithmetical numbers, which included natural numbers, whole numbers, fractional numbers, etc. and fundamental operations on those numbers. In this lesson, we shall introduce algebraic numbers and some other basic concepts of algebra like constants, variables, algebraic expressions, special algebraic expressions, called polynomials and four fundamental operations on them.

OBJECTIVES

After studying this lesson, you will be able to

- *identify variables and constants in an expression;*
- *cite examples of algebraic expressions and their terms;*
- *understand and identify a polynomial as a special case of an algebraic expression;*
- *cite examples of types of polynomials in one and two variables;*
- *identify like and unlike terms of polynomials;*
- *determine degree of a polynomial;*
- *find the value of a polynomial for given value(s) of variable(s), including zeros of a polynomial;*
- *perform four fundamental operations on polynomials.*

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of number systems and four fundamental operations.
- Knowledge of other elementary concepts of mathematics at primary and upper primary levels.

3.1 INTRODUCTION TO ALGEBRA

You are already familiar with numbers 0, 1, 2, 3, ..., $\frac{1}{2}, \frac{3}{4}, ... \sqrt{2}, ...$ etc. and operations of addition (+), subtraction (–), multiplication (×) and division (÷) on these numbers. Sometimes, letters called **literal numbers**, are also used as symbols to represent numbers. Suppose we want to say "The cost of one book is twenty rupees".

In arithmetic, we write : The cost of one book = ₹ 20

In algebra, we put it as: the cost of one book in rupees is x. Thus x stands for a number.

Similarly, a, b, c, x, y, z, etc. can stand for number of chairs, tables, monkeys, dogs, cows, trees, etc. The use of letters help us to think in more general terms.

Let us consider an example, you know that if the side of a square is 3 units, its perimeter is 4 × 3 units. In algebra, we may express this as

$$p = 4\,s$$

where p stands for the number of units of perimeter and s those of a side of the square.

On comparing the language of arithmetic and the language of algebra we find that the language of algebra is

(a) more precise than that of arithmetic.

(b) more general than that of arithmetic.

(c) easier to understand and makes solutions of problems easier.

A few more examples in comparative form would confirm our conclusions drawn above:

Verbal statement	Algebraic statement
(i) A number increased by 3 gives 8	$a + 3 = 8$
(ii) A number increased by itself gives 12	$x + x = 12$, written as $2x = 12$
(iii) Distance = speed × time	$d = s \times t$, written as $d = st$
(iv) A number, when multiplied by itself and added to 5 gives 9	$b \times b + 5 = 9$, written as $b^2 + 5 = 9$
(v) The product of two successive natural numbers is 30	$y \times (y + 1) = 30$, wrtten as $y\,(y + 1) = 30$, where y is a natural number.

Since literal numbers are used to represent numbers of arithmetic, symbols of operation +, –, × and ÷ have the same meaning in algebra as in arithmetic. Multiplication symbols in algebra are often omitted. Thus for 5 × a we write 5a and for a × b we write ab.

3.2 VARIABLES AND CONSTANTS

Consider the months — January, February, March, ..., December of the year 2009. If we represent 'the year 2009' by a and 'a month' by x we find that in this situation 'a' (year 2009) is a fixed entity whereas x can be any one of January, February, March, ..., December. Thus, x is not fixed. It varies. We say that in this case 'a' is a **constant** and 'x' is a **variable**.

Similarly, when we consider students of class X and represent class X by, say, b and a student by, say, y; we find that in this case b (class X) is fixed and so b is a constant and y (a student) is a variable as it can be any one student of class X.

Let us consider another situation. If a student stays in a hostel, he will have to pay fixed room rent, say, ₹ 1000. The cost of food, say ₹ 100 per day, depends on the number of days he takes food there. In this case room rent is constant and the number of days, he takes food there, is variable.

Now think of the numbers.

$$4, -14, \sqrt{2}, \frac{\sqrt{3}}{2}, -\frac{4}{15}, 3x, \frac{21}{8}y, \sqrt{2}z$$

You know that $4, -14, \sqrt{2}, \frac{\sqrt{3}}{2}$, and $-\frac{4}{15}$ are real numbers, each of which has a fixed value while $3x, \frac{21}{8}y$ and $\sqrt{2}z$ contain unknown x, y and z respectively and therefore do not have fixed values like 4, –14, etc. Their values depend on x, y and z respectively. Therefore, x, y and z are variables.

Thus, *a variable is literal number which can have different values whereas a constant has a fixed value.*

In algebra, we usually denote constants by a, b, c and variables x, y, z. However, the context will make it clear whether a literal number has denoted a constant or a variable.

3.3 ALBEGRAIC EXPRESSIONS AND POLYNOMIALS

Expressions, involving arithmetical numbers, variables and symbols of operations are called algebraic expressions. Thus, $3+8$, $8x+4$, $5y$, $7x-2y+6$, $\frac{1}{\sqrt{2x}}, \frac{x}{\sqrt{y-2}}, \frac{ax+by+cz}{x+y+z}$ are all algebraic expressions. You may note that 3 + 8 is both an arithmetic as well as algebraic expression.

An algebraic expression is a combination of numbers, variables and arithmetical operations.

One or more signs + or – separates an algebraic expression into several parts. Each part along with its sign is called a **term** of the expression. Often, the plus sign of the first term is omitted in writing an algebraic expression. For example, we write $x - 5y + 4$ instead of writing $+ x - 5y + 4$. Here x, $-5y$ and 4 are the three terms of the expression.

In $\frac{1}{3}xy$, $\frac{1}{3}$ is called the numerical coefficient of the term and also of xy. **coefficient** of x is $\frac{1}{3}y$ and that of y is $\frac{1}{3}x$. When the numerical coefficient of a term is +1 or –1, the '1' is usually omitted in writing. Thus, numerical coefficent of a term, say, x^2y is +1 and that of $-x^2y$ is –1.

An algebraic expression, in which variable(s) does (do) not occur in the denominator, exponents of variable(s) are whole numbers and numerical coefficients of various terms are real numbers, is called a polynomial.

In other words,

(i) No term of a polynomial has a variable in the denominator;

(ii) In each term of a polynomial, the exponents of the variable(s) are non-negative integers; and

(iii) Numerical coefficient of each term is a real number.

Thus, for example, 5, $3x - y$, $\frac{1}{3}a - b + \frac{7}{2}$ and $\frac{1}{4}x^3 - 2y^2 + xy - 8$ are all polynomials

whereas $x^3 - \frac{1}{x}$, $\sqrt{x+y}$ and $x^{\frac{2}{3}} + 5$ are not polynomials.

$x^2 + 8$ is a polynomial in one variable x and $2x^2 + y^3$ is a polynomial in two variables x and y. In this lesson, we shall restrict our discussion of polynomials including two variables only.

General form of a polynomial in one variable x is:

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

where coefficients $a_0, a_1, a_2, \ldots a_n$ are real numbers, x is a variable and n is a whole number. $a_0, a_1x, a_2x^2, \ldots, a_nx^n$ are $(n + 1)$ terms of the polynomial.

An algebraic expression or a polynomial, consisting of only one term, is called a **monomial**.

Thus, -2, $3y$, $-5x^2$, xy, $\frac{1}{2}x^2y^3$ are all monomials.

An algebraic expression or a polynomial, consisting of only two terms, is called a **binomial**. Thus, $5 + x$, $y^2 - 8x$, $x^3 - 1$ are all bionomials.

An algebraic expression or a polynomial, consisting of only three terms, is called a **trinomial**. Thus $x + y + 1$, $x^2 + 3x + 2$, $x^2 + 2xy + y^2$ are all trinomials.

The terms of a polynomial, having the same variable(s) and the same exponents of the variable(s), are called like terms.

For example, in the expression

$$3xy + 9x + 8xy - 7x + 2x^2$$

the terms $3xy$ and $8\,xy$ are like terms; also $9x$ and $-7x$ are **like terms** whereas $9x$ and $2x^2$ are not like terms. Terms that are not like, are called **unlike terms**. In the above expression $3xy$ and $-7x$ are also unlike terms.

Note that arithmetical numbers are like terms. For example, in the polynomials $x^2 + 2x + 3$ and $x^3 - 5$, the terms 3 and -5 are regrded as like terms since $3 = 3x^0$ and $-5 = -5x^0$.

The terms of the expression

$$2x^2 - 3xy + 9y^2 - 7y + 8$$

are all unlike, i.e., there are no two like terms in this expression.

Example 3.1: Write the variables and constants in $2x^2y + 5$.

Solution: Variables : x and y

Constants: 2 and 5

Example 3.2: In $8x^2y^3$, write the coefficient of

(i) x^2y^3 (ii) x^2 (iii) y^3

Solution: (i) $8x^2y^3 = 8 \times (x^2y^3)$

$\therefore$ Coefficient of x^2y^3 is 8

(ii) $8x^2y^3 = 8y^3 \times (x^2)$

$\therefore$ Coefficient of x^2 is $8y^3$.

(iii) $8x^2y^3 = 8x^2 \times (y^3)$

$\therefore$ Coefficient of y^3 is $8x^2$.

Example 3.3: Write the terms of expression

$$3x^2y - \frac{5}{2}x - \frac{1}{3}y + 2$$

Solution: The terms of the given expression are

$$3x^2y,\ -\frac{5}{2}x,\ -\frac{1}{3}y,\ 2$$

Example 3.4: Which of the following algebraic expressions are polynomials?

(i) $\frac{1}{2}+x^3-2x^2+\sqrt{6}x$ (ii) $x+\frac{1}{x}$

(iii) $2x^2+3x-5\sqrt{x}+6$ (iv) $5-x-x^2-x^3$

Solution: (i) and (iv) are polynomials.

In (ii), second term is $\frac{1}{x}=x^{-1}$. Since second term contains negative exponent of the variable, the expression is not a polynomial.

In (iii), third term is $-5\sqrt{x}=-5x^{\frac{1}{2}}$. Since third term contains fractional exponent of the variable, the expression is not a polynomial.

Example 3.5: Write like terms, if any, in each of the following expressions:

(i) $x+y+2$ (ii) $x^2-2y-\frac{1}{2}x^2+\sqrt{3}y-8$

(iii) $1-2xy+2x^2y-2xy^2+5x^2y^2$ (iv) $\frac{2}{\sqrt{3}}y-\frac{1}{3}z+\frac{\sqrt{5}}{3}y+\frac{1}{3}$

Solution: (i) There are no like terms in the expression.

(ii) x^2 and $-\frac{1}{2}x^2$ are like terms, also $-2y$ and $\sqrt{3}\,y$ are like terms

(iii) There are no like terms in the expression.

(iv) $\frac{2}{\sqrt{3}}y$ and $\frac{\sqrt{5}}{3}y$ are like terms

CHECK YOUR PROGRESS 3.1

1. Write the variables and constants in each of the following:

(i) $1+y$ (ii) $\frac{2}{3}x+\frac{1}{3}y+7$ (iii) $\frac{4}{5}x^2y^3$

(iv) $\frac{2}{5}xy^5+\frac{1}{2}$ (v) $2x^2+y^2-8$ (vi) $x+\frac{1}{x}$

2. In $2x^2y$, write the coefficient of

   (i) x^2y (ii) x^2 (iii) y

3. Using variables and operation symbols, express each of the following verbal statements as algebraic statements:

   (i) three less than a number equals fifteen.

   (ii) A number increased by five gives twenty-two.

4. Write the terms of each of the following expressions:

   (i) $2 + abc$ (ii) $a + b + c + 2$ (iii) $x^2y - 2xy^2 - \frac{1}{2}$

   (iv) $\frac{1}{8}x^3y^2$

5. Identify like terms, if any, in each of the following expressions:

   (i) $-xy^2 + x^2y + y^2 + \frac{1}{3}y^2x$ (ii) $6a + 6b - 3ab + \frac{1}{4}a^2b + ab$

   (iii) $ax^2 + by^2 + 2c - a^2x - b^2y - \frac{1}{3}c^2$

6. Which of the following algebraic expressions are polynomials?

   (i) $\frac{1}{3}x^3 + 1$ (ii) $5^2 - y^2 - 2$ (iii) $4x^{-3} + 3y$

   (iv) $5\sqrt{x+y} + 6$ (v) $3x^2 - \sqrt{2}\,y^2$ (vi) $y^2 - \frac{1}{y^2} + 4$

7. Identify each of the following as a monomial, binomial or a trinomial:

   (i) $x^3 + 3$ (ii) $\frac{1}{3}x^3y^3$ (iii) $2y^2 + 3yz + z^2$

   (iv) $5 - xy - 3x^2y^2$ (v) $7 - 4x^2y^2$ (vi) $-8x^3y^3$

3.4 DEGREE OF A POLYNOMIAL

The sum of the exponents of the variables in a term is called the **degree** of that term. For example, the degree of $\frac{1}{2}x^2y$ is 3 since the sum of the exponents of x and y is 2 + 1, i.e., 3. Similarly, the degree of the term $2x^5$ is 5. The degree of a non-zero constant, say, 3 is 0 since it can be written as $3 = 3 \times 1 = 3 \times x^0$, as $x^0 = 1$.

A polynomial has a number of terms separated by the signs + or –. **The degree of a polynomial is the same as the degree of its term or terms having the highest degree and non-zero coefficient.**

For example, consider the polynomial

$$3x^4y^3 + 7\,xy^5 - 5x^3y^2 + 6xy$$

It has terms of degrees 7, 6, 5, and 2 respectively, of which 7 is the highest. Hence, the degree of this polynomial is 7.

A polynomial of degree 2 is also called a **quadratic polynomial**. For example, $3 - 5x + 4x^2$ and $x^2 + xy + y^2$ are quadratic polynomials.

Note that **the degree of a non-zero constant polynomial is taken as zero.**

When all the coefficients of variable(s) in the terms of a polynomial are zeros, the polynomial is called a **zero polynomial. The degree of a zero polynomial is not defined.**

3.5 EVALUATION OF POLYNOMIALS

We can evaluate a polynomial for given value of the variable occuring in it. Let us understand the steps involved in evaluation of the polynomial $3x^2 - x + 2$ for $x = 2$. Note that we restrict ourselves to polynomials in one variable.

Step 1: Substitute given value(s) in place of the variable(s).

Here, when $x = 2$, we get $3 \times (2)^2 - 2 + 2$

Step 2: Simplify the numerical expression obtained in Step 1.

$3 \times (2)^2 - 2 + 2 = 3 \times 4 = 12$

Therefore, when $x = 2$, we get $3x^2 - x + 2 = 12$

Let us consider another example.

Example 3.6: Evaluate

(i) $1 - x^5 + 2x^6 + 7\,x$ for $x = \frac{1}{2}$

(ii) $5x^3 + 3x^2 - 4x - 4$ for $x = 1$

Solution: (i) For $x = \frac{1}{2}$, the value of the given polynomial is:

$$= 1 - \left(\frac{1}{2}\right)^5 + 2\left(\frac{1}{2}\right)^6 + 7 \times \frac{1}{2}$$

$$= 1 - \frac{1}{32} + \frac{1}{32} + \frac{7}{2}$$

$$= \frac{9}{2} = 4\frac{1}{2}$$

(ii) For x = 1, the value of the given polynomial is:

$5 \times (1)^3 + 3 \times (1)^2 - 4 \times 1 - 4$

$= 5 + 3 - 4 - 4 = 0$

3.6 ZERO OF A POLYNOMIAL

The value(s) of the variable for which the value of a polynomial in one variable is zero is (are) called **zero(s) of the polynomial**. In Example 3.6(ii) above, the value of the polynomial $5x^3 + 3x^2 - 4x - 4$ for x = 1 is zero. Therefore, we say that x = 1 is a zero of the polynomial $5x^3 + 3x^2 - 4x - 4$.

Let us consider another example.

Example 3.7: Determine whether given value is a zero of the given polynomial:

(i) $x^3 + 3x^2 + 3x + 2$; $x = -1$

(ii) $x^4 - 4x^3 + 6x^2 - 4x + 1$; $x = 1$

Solution: (i) For x = – 1, the value of the given polynomial is

$(-1)^3 + 3 \times (-1)^2 + 3 \times (-1) + 2$

$= -1 + 3 - 3 + 2$

$= 1 \ (\neq 0)$

Hence, x = – 1 is not a zero of the given polynomial.

(ii) For x = 1, the value of the given polynomial is

$(1)^4 - 4 \times (1)^3 + 6 \times (1)^2 - 4 \times 1 + 1$

$= 1 - 4 + 6 - 4 + 1$

$= 0$

Hence, x = 1 is a zero of the given polynomial.

CHECK YOUR PROGRESS 3.2

1. Write the degree of each of the following monomials:

(i) $\frac{18}{5}x^7$ (ii) $\frac{7}{8}y^3$ (iii) 10 x (iv) 27

2. Rewrite the following monomials in increasing order of their degrees:

   $-3x^6, \frac{2}{9}x^2, 9x, -25x^3, 2.5$

3. Determine the degree of each of the following polynomials:

   (i) $5x^6y^4 + 1$ (ii) $10^5 + xy^3$ (iii) $x^2 + y^2$ (iv) $x^2y + xy^2 - 3xy + 4$

4. Evaluate each of the following polynomials for the indicated value of the variable:

   (i) $x^2 - 25$ for $x = 5$ (ii) $x^2 + 3x - 5$ for $x = -2$

   (iii) $\frac{2}{3}x^3 + \frac{4}{5}x^2 - \frac{7}{5}$ for $x = -1$ (iv) $2x^3 - 3x^2 - 3x + 12$ for $x = -2$

5. Verify that each of $x = 2$ and $x = 3$ is a zero of the polynomial $x^2 - 5x + 6$.

3.7 ADDITION AND SUBTRACTION OF POLYNOMIALS

You are now familiar that polynomials may consist of like and unlike terms. In adding polynomials, we add their like terms together. Similarly, in subtracting a polynomial from another polynomial, we subtract a term from a like term. The question, now, arises 'how do we add or subtract like terms?' Let us take an example.

Suppose we want to add like terms 2x and 3x. The procedure, that we follow in arithmetic, we follow in algebra too. You know that

$$5 \times 6 + 5 \times 7 = 5 \times (6 + 7)$$

$$6 \times 5 + 7 \times 5 = (6 + 7) \times 5$$

Therefore, $2x + 3x = 2 \times x + 3 \times x$

$$= (2 + 3) \times x$$

$$= 5 \times x$$

$$= 5x$$

Similarly, $2xy + 4\,xy = (2 + 4)\,xy = 6xy$

$$3x^2y + 8x^2y = (3 + 8)x^2y = 11x^2y$$

In the same way, since

$$7 \times 5 - 6 \times 5 = (7 - 6) \times 5 = 1 \times 5$$

$$\therefore\ 5y - 2y = (5 - 2) \times y = 3y$$

and $9x^2y^2 - 5x^2y^2 = (9 - 5)x^2y^2 = 4x^2y^2$

In view of the above, we conclude:

1. *The sum of two (or more) like terms is a like term whose numerical coefficient is the sum of the numerical coefficients of the like terms.*

2. *The difference of two like terms is a like term whose numerical coefficient is the difference of the numerical coefficients of the like terms.*

Therefore, to add two or more polynomials, we take the following steps:

Step 1: Group the like terms of the given polynomials together.

Step 2: Add the like terms together to get the sum of the given polynomials.

Example 3.8: Add $-3x + 4$ and $2x^2 - 7x - 2$

Solution:

$$\begin{aligned} & (-3x + 4) + (2x^2 - 7x - 2) \\ &= 2x^2 + (-3x - 7x) + (4 - 2) \\ &= 2x^2 + (-3 - 7)x + 2 \\ &= 2x^2 + (-10)x + 2 \\ &= 2x^2 - 10x + 2 \end{aligned}$$

$$\therefore\ (-3x + 4) + (2x^2 - 7x - 2) = 2x^2 - 10x + 2$$

Polynomials can be added more conveniently if

(i) the given polynomials are so arranged that their like terms are in one column, and

(ii) the coefficients of each column (i.e. of the group of like terms) are added

Thus, Example 3.8 can also be solved as follows:

$$\begin{array}{rrr} & -3x & +4 \\ 2x^2 & -7x & -2 \\ \hline 2x^2 & +(-7-3)x & +(4-2) \end{array}$$

$$\therefore\ (-3x + 4) + (2x^2 - 7x - 2) = 2x^2 - 10x + 2$$

Example 3.9: Add $5x + 3y - \frac{3}{4}$ and $-2x + y + \frac{7}{4}$

Solution:

$$\begin{array}{l} 5x + 3y - \frac{3}{4} \\ -2x + y + \frac{7}{4} \\ \hline 3x + 4y + \left(\frac{7}{4} - \frac{3}{4}\right) \\ \hline = 3x + 4y + 1 \end{array}$$

$$\therefore \left(5x + 3y - \frac{3}{4}\right) + \left(-2x + y + \frac{7}{4}\right) = 3x + 4y + 1$$

Example 3.10: Add $\frac{3}{2}x^3 + x^2 + x + 1$ and $x^4 - \frac{x^3}{2} - 3x + 1$

Solution:

$$\frac{3}{2}x^2 + x^2 + x + 1$$

$$+x^4 - \frac{1}{2}x^3 \qquad - 3x + 1$$

$$x^4 + \left(\frac{3}{2} - \frac{1}{2}\right)x^3 + x^2 + (1-3)x + (1+1)$$

$$= x^4 + x^3 + x^2 - 2x + 2$$

$$\therefore \left(\frac{3}{2}x^2 + x^2 + x + 1\right) + \left(x^4 - \frac{x^3}{2} - 3x + 1\right) = x^4 + x^3 + x^2 - 2x + 2$$

In order to subtract one polynomial from another polynomial, we go through the following three steps:

Step 1: Arrange the given polynomials in columns so that like terms are in one column.

Step 2: Change the sign (from + to – and from – to +) of each term of the polynomial to be subtracted.

Step 3: Add the like terms of each column separately.

Let us understand the procedure by means of some examples.

Example 3.11: Subtract $-4x^2 + 3x + \frac{2}{3}$ from $9x^2 - 3x - \frac{2}{7}$.

Solution:

$$9x^2 - 3x - \frac{2}{7}$$

$$-4x^2 + 3x + \frac{2}{3}$$

$$+ \qquad - \qquad -$$

$$(9+4)x^2 + (-3-3)x + \left(-\frac{2}{7} - \frac{2}{3}\right)$$

$$= 13x^2 - 6x - \frac{20}{21}$$

$$\therefore \left(9x^2 - 3x - \frac{2}{7}\right) - \left(-4x^2 + 3x + \frac{2}{3}\right) = 13x^2 - 6x - \frac{20}{21}$$

Example 3.12: Subtract $3x - 5x^2 + 7 + 3x^3$ from $2x^2 - 5 + 11x - x^3$.

Solution:

$$\begin{array}{l} -x^3 + 2x^2 + 11x - 5 \\ 3x^3 \ - 5x^2 + 3x \ \ + 7 \\ - \quad + \quad - \quad\quad - \\ \hline (-1-3)x^3 + (2+5)x^2 + (11-3)x + (-5-7) \\ \hline = -4x^3 + 7x^2 + 8x - 12 \end{array}$$

$\therefore (2x^2 - 5 + 11x - x^3) - (3x - 5x^2 + 7 + 3x^3) = -4x^3 + 7x^2 + 8x - 12$

Example 3.13: Subtract $12xy - 5y^2 - 9x^2$ from $15xy + 6y^2 + 7x^2$.

Solution:

$$\begin{array}{l} 15xy + 6y^2 + 7x^2 \\ 12xy - 5y^2 - 9x^2 \\ - \quad\ + \quad\ + \\ \hline 3xy + 11y^2 + 16x^2 \\ \hline \end{array}$$

Thus, $(15xy + 6y^2 + 7x^2) - (12xy - 5y^2 - 9x^2) = 3xy + 11y^2 + 16x^2$

We can also directly subtract without arranging expressions in columns as follows:

$(15xy + 6y^2 + 7x^2) - (12xy - 5y^2 - 9x^2)$

$= 15xy + 6y^2 + 7x^2 - 12xy + 5y^2 + 9x^2$

$= 3xy + 11y^2 + 16x^2$

In the same manner, we can add more than two polynomials.

Example 3.14: Add polynomials $3x + 4y - 5x^2$, $5y + 9x$ and $4x - 17y - 5x^2$.

Solution:

$$\begin{array}{l} 3x + 4y - 5x^2 \\ 9x + 5y \\ 4x - 17y - 5x^2 \\ \hline 16x - 8y - 10x^2 \\ \hline \end{array}$$

$\therefore (3x + 4y - 5x^2) + (5y + 9x) + (4x - 17y - 5x^2) = 16x - 8y - 10x^2$

Example 3.15: Subtract $x^2 - x - 1$ from the sum of $3x^2 - 8x + 11$, $-2x^2 + 12x$ and $-4x^2 + 17$.

Solution: Firstly we find the sum of $3x^2 - 8x + 11, -2x^2 + 12x$ and $-4x^2 + 17$.

$$\begin{array}{rrr} 3x^2 & -\ 8x & +\ 11 \\ -\ 2x^2 & +\ 12x & \\ -\ 4x^2 & & +\ 17 \\ \hline -\ 3x^2 & +\ 4x & +\ 28 \\ \hline \end{array}$$

Now, we subtract $x^2 - x - 1$ from this sum.

$$\begin{array}{rrr} -\ 3x^2 & +\ 4x & +\ 28 \\ x^2 & -\ x & -\ 1 \\ - & + & + \\ \hline -\ 4x^2 & +\ 5x & +\ 29 \\ \hline \end{array}$$

Hence, the required result is $-4x^2 + 5x + 29$.

CHECK YOUR PROGRESS 3.3

1. Add the following pairs of polynomials:

   (i) $\frac{2}{3}x^2 + x + 1; \quad \frac{3}{7}x^2 + \frac{1}{4}x + 5$

   (ii) $\frac{7}{5}x^3 - x^2 + 1; \quad 2x^2 + x - 3$

   (iii) $7x^2 - 3x + 4y; \quad 3x^3 + 5x^2 - 4x + \frac{7}{3}y$

   (iv) $2x^3 + 7x^2y - 5xy + 7; \ -2x^2y + 7x^3 - 3xy - 7$

2. Add:

   (i) $x^2 - 3x + 5, \ 5 + 7x - 3x^2$ and $x^2 + 7$

   (ii) $\frac{1}{3}x^2 + \frac{7}{8}x - 5, \ \frac{2}{3}x^2 + 5 + \frac{1}{8}x$ and $-x^2 - x$

   (iii) $a^2 - b^2 + ab, \ b^2 - c^2 + bc$ and $c^2 - a^2 + ca$

   (iv) $2a^2 + 3b^2, 5a^2 - 2b^2 + ab$ and $-6a^2 - 5ab + b^2$

3. Subtract:

   (i) $7x^3 - 3x^2 + 2$ from $x^2 - 5x + 2$

   (ii) $3y - 5y^2 + 7 + 3y^3$ from $2y^2 - 5 + 11y - y^3$

(iii) $2z^3 + 7z - 5z^2 + 2$ from $5z + 7 - 3z^2 + 5z^3$

(iv) $12x^3 - 3x^2 + 11x + 13$ from $5x^3 + 7x^2 + 2x - 4$

4. Subtract $4a - b - ab + 3$ from the sum of $3a - 5b + 3ab$ and $2a + 4b - 5ab$.

3.8 MULTIPLICATION OF POLYNOMIALS

To multiply a monomial by another monomial, we make use of laws of exponents and the rule of signs. For example,

$$3a \times a^2b^2c^2 = (3 \times 1)\, a^{2+1}\, b^2\, c^2 = 3a^3b^2c^2$$

$$-5x \times 2\,xy^3 = (-5 \times 2)\, x^{1+1}\, y^3 = -10\, x^2y^3$$

$$-\frac{1}{2}y^2z \times \left(-\frac{1}{3}\right)yz = \left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)y^{2+1}z^{1+1} = \frac{1}{6}y^3z^2$$

To multiply a polynomial by a monomial, we multiply each term of the polynomial by the monomial. For example

$$x^2y \times (-y^2 + 2xy + 1) = x^2y \times (-y^2) + (x^2y) \times 2xy + (x^2y) \times 1$$

$$= -x^2y^3 + 2x^3y^2 + x^2y$$

To multiply a polynomial by another polynomial, we multiply each term of one polynomial by each term of the other polynomial and simplify the result by combining the terms. It is advisable to arrange both the polynomials in increasing or decreasing powers of the variable. For example,

$$(2n + 3)(n^2 - 3n + 4) = 2n \times n^2 + 2n \times (-3n) + 2n \times 4 + 3 \times n^2 + 3 \times (-3n) + 3 \times 4$$

$$= 2n^3 - 6n^2 + 8n + 3n^2 - 9n + 12$$

$$= 2n^3 - 3n^2 - n + 12$$

Let us take some more examples.

Example 3.16: Find the product of $(0.2x^2 + 0.7\,x + 3)$ and $(0.5\,x^2 - 3x)$

Solution: $(0.2x^2 + 0.7\,x + 3) \times (0.5\,x^2 - 3x)$

$$= 0.2x^2 \times 0.5\,x^2 + 0.2x^2 \times (-3x) + 0.7\,x \times 0.5\,x^2 + 0.7\,x \times (-3x) + 3 \times 0.5x^2 + 3 \times (-3x)\,)$$

$$= 0.1x^4 - 0.60x^3 + 0.35x^3 - 2.1x^2 + 1.5\,x^2 - 9x$$

$$= 0.1\,x^4 - 0.25\,x^3 - 0.6x^2 - 9x$$

Example 3.17: Multiply $2x - 3 + x^2$ by $1 - x$.

Solution: Arranging polynomials in decreasing powers of x, we get

$$\begin{aligned}(x^2 + 2x - 3) \times (-x + 1) &= x^2 \times (-x) + x^2 \times (1) + 2x \times (-x) + 2x \times 1 - 3 \times (-x) \\ &\quad - 3 \times 1 \\ &= -x^3 + x^2 - 2x^2 + 2x + 3x - 3 \\ &= -x^3 - x^2 + 5x - 3\end{aligned}$$

Alternative method:

$$\begin{array}{rrrrl} & x^2 & +2x & -3 & \leftarrow \text{one polynomial} \\ & & -x & +1 & \leftarrow \text{other polynomial} \\ \hline -x^3 & -2x^2 & +3x & & \\ & +x^2 & +2x & -3 & \leftarrow \text{Partial products} \\ \hline -x^3 & -x^2 & +5x & -3 & \leftarrow \text{Product} \\ \hline \end{array}$$

3.9 DIVISION OF POLYNOMIALS

To divide a monomial by another monomial, we find the quotient of numerical coefficients and variable(s) separately using laws of exponents and then multiply these quotients. For example,

(i) $$25x^3y^3 \div 5x^2y = \frac{25x^3y^3}{5x^2y} = \frac{25}{5} \times \frac{x^3}{x^2} \times \frac{y^3}{y}$$
$$= 5 \times x^1 \times y^2$$
$$= 5xy^2$$

(ii) $$-12ax^2 \div 4x = -\frac{12ax^2}{4x} = \frac{-12}{4} \times \frac{a}{1} \times \frac{x^2}{x}$$
$$= -3ax$$

To divide a polynomial by a monomial, we divide each term of the polynomial by the monomial. For example,

(i) $$(15x^3 - 3x^2 + 18x) \div 3x = \frac{15x^3}{3x} - \frac{3x^2}{3x} + \frac{18x}{3x}$$
$$= 5x^2 - x + 6$$

(ii) $$(-8x^2 + 10x) \div (-2x) = \frac{-8x^2}{-2x} + \frac{10x}{-2x}$$
$$= \left(\frac{-8}{-2}\right)\left(\frac{x^2}{x}\right) + \frac{10}{(-2)} \times \frac{x}{x}$$
$$= 4x - 5$$

The process of division of a polynomial by another polynomial is done on similar lines as in arithmetic. Try to recall the process when you divided 20 by 3.

$$\begin{array}{r|l} & 6 \leftarrow \text{Quotient} \\ \text{Divisor} \rightarrow 3 & 20 \leftarrow \text{Dividend} \\ & 18 \\ \hline & 2 \leftarrow \text{Remainder} \end{array}$$

The steps involved in the process of division of a polynomial by another polynomial are explained below with the help of an example.

Let us divide $2x^2 + 5x + 3$ by $2x + 3$.

Step 1: Arrange the terms of both the polynomials in decreasing powers of the variable common to both the polynomials.

$$2x+3\overline{)2x^2+5x+3}$$

Step 2: Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.

$$\begin{array}{r}x\\2x+3\overline{)2x^2+5x+3}\end{array}$$

Step 3: Multiply all the terms of the divisor by the first term of the quotient and subtract the result from the dividend, to obtain a remainder (as next dividend)

Step 4: Divide the first term of the resulting dividend by the first term of the divisor and write the result as the second term of the quotient.

$$\begin{array}{r}x\\2x+3\overline{)2x^2+5x+3}\\2x^2+3x\\-\quad\;-\\\hline 2x+3\end{array}$$

Step 5: Multiply all the terms of the divisor by the second term of the quotient and subtract the result from the resulting dividend of Step 4.

Step 6: Repeat the process of Steps 4 and 5, till you get either the remainder zero or a polynomial having the highest exponent of the variable lower than that of the divisor.

$$\begin{array}{r}x+1\\2x+3\overline{)2x^2+5x+3}\\2x^2+3x\\-\quad\;-\\\hline 2x+3\\2x+3\\-\quad-\\\hline 0\\\hline\end{array}$$

In the above example, we got the quotient $x + 1$ and remainder 0.

Let us now consider some more examples.

Example 3.18 : Divide $x^3 - 1$ by $x - 1$.

Solution:

$$\begin{array}{r}x^2+x+1\\x-1\overline{)x^3-1}\\x^3-x^2\\-\quad+\;\;\\\hline x^2-1\\x^2-x\\-\quad+\\\hline x-1\end{array}$$

$$\begin{array}{r} x - 1 \\ - \quad + \\ \hline 0 \\ \hline \end{array}$$

We get quotient $x^2 + x + 1$ and remainder 0.

Example 3.19: Divide $5x - 11 - 12x^2 + 2x^3$ by $2x - 5$.

Solution: Arranging the dividend in decreasing powers of x, we get it as

$2x^3 - 12x^2 + 5x - 11$

So,

$$\begin{array}{r|l} & x^2 - \frac{7}{2}x - \frac{25}{4} \\ \hline 2x - 5 & 2x^3 - 12x^2 + 5x - 11 \\ & 2x^3 - 5x^2 \\ & - \quad + \\ \hline & -7x^2 + 5x - 11 \\ & -7x^2 + \frac{35}{2}x \\ & + \quad - \\ \hline & -\frac{25}{2}x - 11 \\ & -\frac{25}{2}x + \frac{125}{4} \\ & + \quad - \\ \hline & -\frac{169}{4} \\ \hline \end{array}$$

We get quotient $x^2 - \frac{7}{2}x - \frac{25}{4}$ and remainder $-\frac{169}{4}$.

CHECK YOUR PROGRESS 3.4

1. Multiply:

   (i) $9b^2c^2$ by $3b$ (ii) $5x^3y^5$ by $-2xy$

   (iii) $2xy + y^2$ by $-5x$ (iv) $x + 5y$ by $x - 3y$

2. Write the quotient:

   (i) $x^5y^3 \div x^2y^2$ (ii) $-28y^7z^2 \div (-4y^3z^2)$

   (iii) $(a^4 + a^3b^5) \div a^2$ (iv) $-15b^5c^6 \div 3b^2c^4$

3. Divide and write the quotient and the remainder:

   (i) $x^2 - 1$ by $x + 1$ (ii) $x^2 - x + 1$ by $x + 1$

   (iii) $6x^2 - 5x + 1$ by $2x - 1$ (iv) $2x^3 + 4x^2 + 3x + 1$ by $x + 1$

LET US SUM UP

- A literal number (unknown quantity), which can have various values, is called a variable.
- A constant has a fixed value.
- An algebraic expression is a combination of numbers, variables and arithmetical operations. It has one or more terms joined by the signs + or –.
- Numerical coefficient of a term, say, 2xy is 2. Coefficient of x is 2y and that of y is 2x.
- Numerical coefficient of non-negative x is + 1 and that of – x is – 1.
- An algebraic expression, in which variable(s) does (do) not occur in the denominator, exponents of variables are whole numbers and numerical coefficients of various terms are real numbers, is called a polynomial.
- The standard form of a polynomial in one variable x is:

  $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ (or written in reverse order) where $a_0, a_1, a_2, ... a_n$ are real numbers and n, n–1, n–2, ..., 3, 2, 1 are whole numbers.
- An algebraic expression or a polynomial having one term is called a monomial, that having two terms a bionomial and the one having three terms a trinomial.
- The terms of an algebraic expression or a polynomial having the same variable(s) and same exponent(s) of variable(s) are called like terms. The terms, which are not like, are called unlike terms.
- The sum of the exponents of variables in a term is called the degree of that term.
- The degree of a polynomial is the same as the degree of its term or terms having the highest degree and non-zero numerical coefficient.
- The degree of a non-zero constant polynomial is zero.
- The process of substituting a numerical value for the variable(s) in an algebraic expression (or a polynomial) is called evaluation of the algebraic expression (or polynomial).
- The value(s) of variable(s), for which the value of a polynomial is zero, is (are) called zero(s) of the polynomial.
- The sum of two like terms is a like term whose numerical coefficient is the sum of the numerical coefficients of the two like terms.

- The difference of two like terms is a like term whose numerical coefficient is the difference of the numerical coefficients of the two like terms.
- To multiply or divide a polynomial by a monomial, we multiply or divide each term of the polynomial separately using laws of exponents and the rule of signs.
- To multiply a polynomial by a polynomial, we multiply each term of one polynomial by each term of the other polynomial and simplify the result by combining like terms.
- To divide a polynomial by a polynomial, we usually arrange the terms of both the polynomials in decreasing powers of the variable common to both of them and take steps of division on similar lines as in arithmetic in case of numbers.

TERMINAL EXERCISE

1. Mark a tick (✓) against the correct alternative:

(i) The coefficient of x^4 in $6x^4y^2$ is

(A) 6 (B) y^2 (C) $6y^2$ (D) 4

(ii) Numerical coefficient of the monomial $-x^2y^4$ is

(A) 2 (B) 6 (C) 1 (D) –1

(iii) Which of the following algebraic expressions is a polynomial?

(A) $\frac{1}{\sqrt{2}}x^2 - \sqrt{8} + 3.7x$ (B) $2x + \frac{1}{2x} - 4$

(C) $(x^2 - 2y^2) \div (x^2 + y^2)$ (D) $6 + \sqrt{x} - x - 15x^2$

(iv) How many terms does the expression $1 - \sqrt{2}a^2b^3 - (7a)(2b) + \sqrt{3}b^2$ contain?

(A) 5 (B) 4 (C) 3 (D) 2

(v) Which of the following expressions is a binomial?

(A) $2x^2y^2$ (B) $x^2 + y^2 - 2xy$

(C) $2 + x^2 + y^2 + 2x^2y^2$ (D) $1 - 3xy^3$

(vi) Which of the following pairs of terms is a pair of like terms?

(A) 2a, 2b (B) $2xy^3, 2x^3y$

(C) $3x^2y, \frac{1}{\sqrt{2}}yx^2$ (D) 8, 16 a

(vii) A zero of the polynomial $x^2 - 2x - 15$ is

(A) $x = -5$ (B) $x = -3$

(C) $x = 0$ (D) $x = 3$

(viii) The degree of the polynomial $x^3y^4 + 9x^6 - 8y^5 + 17$ is

(A) 7 (B) 17

(C) 5 (D) 6

2. Using variables and operation symbols, express each of the following verbal statements as algebraic statement:

   (i) A number added to itself gives six.

   (ii) Four subtracted from three times a number is eleven.

   (iii) The product of two successive odd numbers is thirty-five.

   (iv) One-third of a number exceeds one-fifth of the number by two.

3. Determine the degree of each of the following polynomials:

   (i) 3^{27} (ii) $x + 7x^2y^2 - 6xy^5 - 18$ (iii) $a^4x + bx^3$ where a and b are constants

   (iv) $c^6 - a^3x^2y^2 - b^2x^3y$ Where a, b and c are constants.

4. Determine whether given value is a zero of the polynomial:

   (i) $x^2 + 3x - 40$; $x = 8$

   (ii) $x^6 - 1$; $x = -1$

5. Evaluate each of the following polynomials for the indicated value of the variable:

   (i) $2x - \frac{3}{2}x^2 + \frac{4}{5}x^5 + 7x^3$ at $x = \frac{1}{2}$

   (ii) $\frac{4}{5}y^3 + \frac{1}{5}y^2 - 6y - 65$ at $y = -5$

6. Find the value of $\frac{1}{2}n^2 + \frac{1}{2}n$ for $n = 10$ and verify that the result is equal to the sum of first 10 natural numbers.

7. Add:

   (i) $\frac{7}{3}x^3 + \frac{2}{5}x^2 - 3x + \frac{7}{5}$ and $\frac{2}{3}x^3 + \frac{3}{5}x^2 - 3x + \frac{3}{5}$

   (ii) $x^2 + y^2 + 4xy$ and $2y^2 - 4xy$

   (iii) $x^3 + 6x^2 + 4xy$ and $7x^2 + 8x^3 + y^2 + y^3$

   (iv) $2x^5 + 3x + \frac{2}{3}$ and $-3x^5 + \frac{2}{5}x - 3$

8. Subtract

   (i) $-x^2 + y^2 - xy$ from 0

   (ii) $a + b - c$ from $a - b + c$

   (iii) $x^2 - y^2x + y$ from $y^2x - x^2 - y$

   (iv) $-m^2 + 3mn$ from $3m^2 - 3mn + 8$

9. What should be added to $x^2 + xy + y^2$ to obtain $2x^2 + 3xy$?

10. What should be subtracted from $-13x + 5y - 8$ to obtain $11x - 16y + 7$?

11. The sum of two polynomials is $x^2 - y^2 - 2xy + y - 7$. If one of them is $2x^2 + 3y^2 - 7y + 1$, find the other.

12. If $A = 3x^2 - 7x + 8$, $B = x^2 + 8x - 3$ and $C = -5x^2 - 3x + 2$, find $B + C - A$.

13. Subtract $3x - y - xy$ from the sum of $3x - y + 2xy$ and $-y - xy$. What is the coefficient of x in the result?

14. Multiply

   (i) $a^2 + 5a - 6$ by $2a + 1$ (ii) $4x^2 + 16x + 15$ by $x - 3$

   (iii) $a^2 - 2a + 1$ by $a - 1$ (iv) $a^2 + 2ab + b^2$ by $a - b$

   (v) $x^2 - 1$ by $2x^2 + 1$ (vi) $x^2 - x + 1$ by $x + 1$

   (vii) $x^2 + \frac{2}{3}x + \frac{5}{6}$ by $x - \frac{7}{4}$ (viii) $\frac{2}{3}x^2 + \frac{5}{4}x - 3$ by $3x^2 + 4x + 1$

15. Subtract the product of $(x^2 - xy + y^2)$ and $(x + y)$ from the product of $(x^2 + xy + y^2)$ and $(x - y)$.

16. Divide

   (i) $8x^3 + y^3$ by $2x + y$ (ii) $7x^3 + 18x^2 + 18x - 5$ by $3x + 5$

   (iii) $20x^2 - 15x^3y^6$ by $5x^2$ (iv) $35a^3 - 21a^4b$ by $(-7a^3)$

   (v) $x^3 - 3x^2 + 5x - 8$ by $x - 2$ (vi) $8y^2 + 38y + 35$ by $2y + 7$

   In each case, write the quotient and remainder.

ANSWERS TO CHECK YOUR PROGRESS

3.1

1. (i) y; 1 (ii) x, y; $\frac{2}{3}, \frac{1}{3}, 7$ (iii) x, y; $\frac{4}{5}$

(iv) x, y; $\frac{2}{5}, \frac{1}{2}$ (v) x, y; 2, –8 (vi) x; None

2. (i) 2 (ii) 2y (iii) $2x^2$

3. (i) x – 3 = 15 (ii) x + 5 = 22

4. (i) 2, abc (ii) a,b,c, 2 (iii) $x^2y, -2xy^2, -\frac{1}{2}$

(iv) $\frac{1}{8}x^3y^2$

5. (i) $-xy^2, +\frac{1}{3}y^2x$ (ii) –3ab, + ab (iii) No like terms

6. (i) , (ii) and (v) 7. Monomials (ii) and (vi);

Binomials: (i) and (v); Trinomials : (iii) and (iv)

3.2

1. (i) 7 (ii) 3 (iii) 1 (iv) 0

2. 2.5, 9x, $\frac{2}{9}x^2$, $-25x^3$, $-3x^6$

3. (i) 10 (ii) 4 (iii) 2 (iv) 3

4. (i) 0 (ii) –7 (iii) $-\frac{19}{15}$ (iv) 6

3.3

1. (i) $\frac{23}{11}x^2 + \frac{5}{4}x + 6$ (ii) $\frac{7}{5}x^3 + x^2 + x - 2$

(iii) $3x^3 + 12x^2 - 7x + \frac{19}{3}y$ (iv) $9x^3 + 5x^2y - 8xy$

2. (i) $-x^2 + 4x + 17$ (ii) 0

(iii) ab + bc +ca (iv) $a^2 + 2b^2 - 4ab$

3. (i) $-7x^3 + 4x^2 - 5x$ (ii) $-4y^3 + 7y^2 + 8y - 12$

(iii) $3z^3 + 2z^2 - 2z + 5$ (iv) $-7x^3 + 10x^2 - 9x - 17$

4. a – ab – 3

3.4

1. (i) $27b^3c^2$ (ii) $-10x^4y^6$
   (iii) $-10x^2y - 5xy^2$ (iv) $x^2 + 2xy - 15y^2$
2. (i) x^3y (ii) $7y^4$ (iii) $a^2 + ab^5$ (iv) $-5b^3c^2$
3. (i) $x - 1; 0$ (ii) $x - 2; 3$ (iii) $3x - 1; 0$ (iv) $2x^2 + 2x + 1; 0$

ANSWERS TO TERMINAL EXERCISE

1. (i) C (ii) D (iii) A (iv) B (v) D (vi) C (vii) B (viii) A
2. (i) $y + y = 6$ (ii) $3y - 4 = 11$ (iii) $z(z + 2) = 35$ (iv) $\frac{x}{3} - \frac{x}{5} = 2$
3. (i) 0 (ii) 6 (iii) 3 (iv) 4
4. (i) No (ii) Yes
5. (i) $\frac{37}{24}$ (ii) 0
6. 55
7. (i) $3x^3 + x^2 - 6x + 2$ (ii) $x^2 + 3y^2$
   (iii) $9x^3 + 13x^2 + 4xy + y^2 + y^3$ (iv) $-x^5 + \frac{17}{5}x - \frac{7}{3}$
8. (i) $x^2 - y^2 + xy$ (ii) $2c - 2b$
   (iii) $2y^2x - 2x^2 - 2y$ (iv) $4m^2 - 6mn + 8$
9. $x^2 + 2xy - y^2$
10. $-24x + 21y - 15$
11. $-x^2 - 4y^2 - 2xy + 8y - 8$
12. $-7x^2 + 12x - 9$
13. $2xy - y; 2y$
14. (i) $2a^3 + 11a^2 - 7a - 6$ (ii) $4x^3 + 4x^2 - 33x - 45$
    (iii) $a^3 - 3a^2 + 3a - 1$ (iv) $a^3 + a^2b - ab^2 - b^3$
    (v) $2x^4 - x^2 - 1$ (vi) $x^3 + 1$
    (vii) $x^3 - \frac{13}{12}x^2 - \frac{x}{3} - \frac{35}{24}$ (viii) $2x^4 + \frac{77}{12}x^3 - \frac{10}{3}x^2 - \frac{43}{4}x - 3$
15. $-2y^3$
16. (i) $4x^2 - 2xy + y^2; 0$ (ii) $9x^2 - 9x + 21; -110$
    (iii) $4 - 3xy^6; 0$ (iv) $-5 + 3ab; 0$
    (iv) $x^2 - x + 3; -2$ (v) $4y + 5; 0$